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Geschiedenis History

Emmy Noether and her Principle on Symmetry and Conservation Laws

This article has a twofold purpose. On the one hand it gives a pedestrian but incomplete exposition of Noether's famous principle on the equivalence of symmetries and conservation laws. On the other hand it discusses the minor appreciation for this work at the time in mathematics and physics, while her work in abstract algebra quickly became highly influential in the development of mathematics.

Symplectic geometry

A symplectic manifold is a pair (M,ω) with M a manifold and ω a closed differential 2-form that is nondegenerate. The basic example of a symplectic manifold is the cotangent bundle $\mathrm{T}^*\mathrm{Q}$ of a manifold Q with symplectic form $\omega = d\mathbf{p}\wedge d\mathbf{q}$. Here $\mathbf{q}$ are local coordinates on the configuration space Q and taking $\mathbf{p} = \partial/\partial\mathbf{q}$ as linear coordinates on the cotangent spaces $\mathrm{T}^*_{\mathbf{q}}\mathrm{Q}$ we get $(\mathbf{q},\mathbf{p})$ as local coordinates on the phase space $\mathrm{T}^*\mathrm{Q}$. Locally any symplectic manifold admits so called canonical coordinates $(\mathbf{q},\mathbf{p})$ in which $\omega = d\mathbf{p}\wedge d\mathbf{q}$. This is a result of Darboux from 1882.

Using the symplectic form ω the differential dH of a function H on M determines a vector field $\mathrm{X_H}$ on M by

$$\mathrm{dH(Y)} = -\omega(\mathrm{X_H},\mathrm{Y})$$

for any vector field Y on M. In canonical coordinates this so called Hamilton field of H becomes

$$\mathrm{X_H} = \frac{\partial \mathrm{H}}{\partial \mathbf{p}}\frac{\partial}{\partial \mathbf{q}} - \frac{\partial \mathrm{H}}{\partial \mathbf{q}}\frac{\partial}{\partial \mathbf{p}}$$

and its integral curves are solutions of

$$\frac{d\mathbf{q}}{dt} = \frac{\partial \mathrm{H}}{\partial \mathbf{p}}, \frac{d\mathbf{p}}{dt} = -\frac{\partial \mathrm{H}}{\partial \mathbf{q}}$$

which are Hamilton's equation for the Hamiltonian system $(\mathrm{M},\omega,\mathrm{H})$.

The Poisson bracket $\{\mathrm{F},\mathrm{G}\}$ of two functions F, G on M is a new function on M defined by

$$\{\mathrm{F},\mathrm{G}\} = \mathrm{X_F(G)} = \omega(\mathrm{X_F},\mathrm{X_G}) = \frac{\partial \mathrm{F}}{\partial \mathbf{p}}\frac{\partial \mathrm{G}}{\partial \mathbf{q}} - \frac{\partial \mathrm{F}}{\partial \mathbf{q}}\frac{\partial \mathrm{G}}{\partial \mathbf{p}}$$

and therefore

$$\frac{d\mathrm{F}}{dt} = \{\mathrm{H},\mathrm{F}\}$$

is just a more abstract form of Hamilton's equation. Hence a function F on M is conserved in the Hamiltonian system $(\mathrm{M},\omega,\mathrm{H})$ if and only if $\{\mathrm{H},\mathrm{F}\} = 0$. Note that the Poisson bracket is bilinear and skew symmetric. Siméon Poisson introduced his bracket in 1809 for the following purpose.

Theorem 2.1. *If both functions* F *and* G *are conserved in the Hamiltonian system* $(\mathrm{M},\omega,\mathrm{H})$ *then the Poisson bracket* $\{\mathrm{F},\mathrm{G}\}$ *is again conserved.*

A clean proof of Poisson's theorem was given by Jacobi in 1862 who showed that

$$[\mathrm{X_F},\mathrm{X_G}] = \mathrm{X}_{\{\mathrm{F},\mathrm{G}\}}$$

with $[\cdot,\cdot]$ the commutator bracket of vector fields. In turn this implies the Jacobi identity

$$\{\{\mathrm{F},\mathrm{G}\},\mathrm{H}\} + \{\{\mathrm{G},\mathrm{H}\},\mathrm{F}\} + \{\{\mathrm{H},\mathrm{F}\},\mathrm{G}\} = 0$$

and Poisson's theorem follows immediately.

This work was a prime example for Sophus Lie in the late 19th century for developing his theory of Lie groups and associated Lie algebras, and their actions on general manifolds. His main motivation remained the exploitation of this Lie theory towards solving differential equations with symmetry. An important result of Lie says that an abstract Lie algebra $\mathfrak{g}$ – just a vector space with a bilinear skew symmetric operation satisfying the Jacobi identity – always occurs as the Lie algebra of left invariant vector fields on an associated (local) Lie group G. Moreover, the correspondence $\mathfrak{g} \leftrightarrow \mathrm{G}$ is essentially a bijection.

Lie theory developed into a subject by itself, independent of

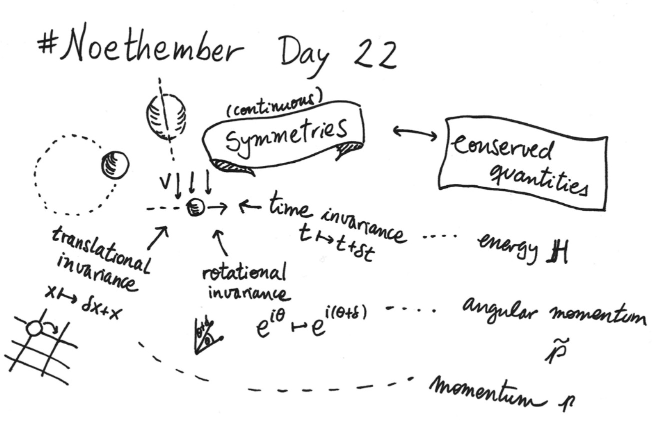

Illustration: Constanza Rojas-Molina

Figure 1 *Noether's principle on symmetry and conservation laws in variational problems.*

potential applicability towards solving differential equations with symmetry. The neat classification of the complex simple Lie algebras by Wilhelm Killing in 1888 and the classification of the real simple Lie algebras in the early 19th century by Élie Cartan and the connection with his theory of symmetric spaces – Riemannian manifolds whose geodesic involutions are isometries – are still landmark results witnessing this development.

The Moment Map

An action $G \times M \to M$ of a Lie group G on a manifold M gives rise to an infinitesimal action $X \mapsto X_M$ of $\mathfrak{g}$ as vector fields on M. If the flow of X_M is given by the action of the one parameter subgroup $t \mapsto \exp(tX)$ of G on M then one should pay attention that

$$[X,Y]_M = -[X_M, Y_M]$$

for all $X, Y \in \mathfrak{g}$. Here the first bracket is the Lie bracket on $\mathfrak{g}$ and the second bracket is the commutator bracket of vector fields on M. In other words, the linear map

$$\mathfrak{g} \to \mathfrak{X}(M),\ X \mapsto X_M$$

is an antihomomorphism of Lie algebras.

Definition 3.1. *An action $G \times M \to M$ of a Lie group G on a symplectic manifold (M,ω) is called Hamiltonian if there exists a map $\mu: M \to \mathfrak{g}^*$ whose coordinate functions $\mu_X(x) = \langle \mu(x), X \rangle$ satisfy*

$$d\mu_X(\cdot) = -\omega(X_M, \cdot),\ \{\mu_X, \mu_Y\} = -\mu_{[X,Y]}$$

for all $X, Y \in \mathfrak{g}$. The map $\mu: M \to \mathfrak{g}^*$ is called the moment (or momentum) map of the Hamiltonian action of G on (M,ω).

The first condition says that X_M is the Hamilton field of the function μ_X while the second condition implies that the moment map μ is equivariant for the Hamiltonian action of G on M and the coadjoint action of G on $\mathfrak{g}^*$. An important example of a Hamiltonian action is given by the following theorem.

Theorem 3.2. *Let $G \times Q \to Q$ be an action of a Lie group G on a smooth manifold Q. Then the induced action $G \times T^*Q \to T^*Q$ on the cotangent bundle $(T^*Q, d\mathbf{p} \wedge d\mathbf{q})$ is Hamiltonian with moment map μ given by*

$$\mu_X(\mathbf{q}, \mathbf{p}) = \langle \mathbf{p}, X_{\mathbf{q}} \rangle$$

with $X \in \mathfrak{g}$ and $X_{\mathbf{q}}$ the value of X_Q at the point $\mathbf{q} \in Q$.

After the early work of Poisson, Jacobi, Lie and Poincaré in the 19th century symplectic geometry went into hybernation. The above definition of moment map dates from around 1970, although in the particular examples of linear momentum linked to translational symmetry and angular momentum linked to rotational symmetry it has been well known much earlier. There were exciting new developments from the 70's on with work by Duistermaat and Guillemin on Weyl's aymptotic law, and by Guillemin and Sternberg, by Atiyah and Bott and by Hitchin on geometric quantization. As a result symplectic geometry became a standard subject taught at masters level, just like algebraic geometry, differential geometry and Lie theory, and with close connections between these topics [1], [9], [2].

Hamilton's Principle of Least Action

Consider a Hamiltonian system

$$(M = T^*Q, \omega = d\mathbf{p} \wedge d\mathbf{q}, H)$$

whose Hamiltonian $H(\mathbf{q},\mathbf{p})$ for all $\mathbf{q} \in Q$ is a strictly convex proper function of $\mathbf{p}$ on $T^*_{\mathbf{q}}Q$. Under this assumption the Legendre transformation of the Hamiltonian function $H(\mathbf{q},\mathbf{p})$ is a Lagrangian function $L(\mathbf{q},\mathbf{v})$ on the tangent bundle TQ defined by

$$L(\mathbf{q},\mathbf{v}) = \langle \mathbf{p}, \mathbf{v} \rangle - H(\mathbf{q},\mathbf{p})$$

with $\mathbf{p}$ the unique solution of the equation $\mathbf{v} = \partial H(\mathbf{q},\mathbf{p})/\partial \mathbf{p}$. For each $\mathbf{q} \in Q$ the Legendre transformation is a diffeomorphism $\mathbf{p} \leftrightarrow \mathbf{v}$ between cotangent space $T^*_{\mathbf{q}}Q$ and tangent space $T_{\mathbf{q}}Q$. Since the Legendre transformation is an involution, we have likewise $\mathbf{p} = \partial L(\mathbf{q},\mathbf{v})/\partial \mathbf{v}$.

The solutions of the variational problem

$$\left[\delta \int_{t_0}^{t_1} L(\mathbf{q},\dot{\mathbf{q}})dt = 0\right]$$

with fixed begin and end points in Q satisfy the Euler–Lagrange equation

$$\left[\frac{d}{dt}\frac{\partial L(\mathbf{q},\dot{\mathbf{q}})}{\partial \dot{\mathbf{q}}} = \frac{\partial L(\mathbf{q},\dot{\mathbf{q}})}{\partial \mathbf{q}}\right].$$

Hamilton's principle of least action says that solutions for the Lagrangian system $(TQ, L(\mathbf{q},\dot{\mathbf{q}}))$ correspond in a bijective manner to solutions of the Hamiltonian system $(T^*Q, d\mathbf{p} \wedge d\mathbf{q}, H(\mathbf{p},\mathbf{q}))$ via the Legendre transformation.

All this is explained well in Chapter 20 of the text book by Ana Cannas da Silva on symplectic geometry [2].

Theorem 4.1. *If the Lagrangian system $(TQ, L(\mathbf{q},\mathbf{v}))$ admits a symmetry with Lie group G and Lie algebra $\mathfrak{g}$ acting on Q then for each $X \in \mathfrak{g}$ we have a first integral $I_X: TQ \to \mathbb{R}$ given by*

$$I_X(\mathbf{q},\mathbf{v}) = \left\langle \frac{\partial L(\mathbf{q},\mathbf{v})}{\partial \mathbf{v}}, X_{\mathbf{q}} \right\rangle$$

with $X_{\mathbf{q}}$ *the value of vector field* X_Q *at* $\mathbf{q} \in Q$.

This formulation of Noether's theorem, together with a direct proof, is given by Arnold in Section 20 of his text book on classical mechanics [1]. But in the special case that the Lagrangian $L(\mathbf{q},\mathbf{v})$ is a strictly convex proper function of $\mathbf{v}$ for all $\mathbf{q} \in Q$, we have $\mathbf{p} = \partial L(\mathbf{q},\mathbf{v})/\partial \mathbf{v}$. Hence, under these assumptions, the above version of Noether's theorem is just an equivalent Lagrangian form of the Hamiltonian results given in Definition 3.1 and Theorem 3.2.

Noether's principle

In her famous paper [10] Emmy Noether considers for $k \in \mathbb{N}$ the more general variational problem

$$\delta \int f(x, u, \frac{\partial u}{\partial x}, \cdots, \frac{\partial^k u}{\partial x^k}) dx = 0$$

with independent variables $x \in \mathbb{R}^n$ and functions $u(x) \in \mathbb{R}^m$ depending on these variables, and the integration performed over compact domains. For such variational problems she subsequently proves two theorems, numbered I and II, about the equivalence of symmetries and conservation laws.

This work of Noether was stimulated by questions of Hilbert and Klein about conservation of energy and momentum in general relativity. Hilbert himself had found a variational reformulation for Einstein's field equation and suggested Noether to look for conservation laws in such a general variational setting. And so she did and found her general results about the equivalence between symmetries and conservation laws in variational problems.

Her paper is by no means easy to read, due to both the general abstract setting and the lack of examples. At the end of her introduction she writes briefly that her "Theorem I includes all known theorems in mechanics about first integrals, while Theorem II can be described as the greatest possible group theoretic generalization of general relativity". I presume that few people tried to read and even fewer understood her paper. In 1924 it was shown by Schouten and Struik that for the Einstein equation the conservation laws coming from symmetries of the coordinates boil down to the contracted Bianchi identity for the Ricci tensor as indicated in Chapter 14 of Einstein's biography by Pais [12]. In the excellent text books on general relativity by Hawking and Ellis [4] and by Misner, Thorne and Wheeler [7] one looks in vain for the name of Emmy Noether.

Noether's paper disappeared for a long time to the edge of oblivion. Apart from the already mentioned difficulty of her paper the interest in Lie theory shifted towards representation theory, stimulated by the recent discovery of quantum mechanics, with beautiful contributions by Heisenberg and Pauli (for the calculation of the Hydrogen spectrum), by Casimir and Van der Waerden (for an algebraic proof of complete reducibility for the simple Lie algebras, which was later simplified by Richard Brauer), and by Wigner for his chain of applications of physical phenomena by group theory (spectral theory for atoms, molecules or crystals with symmetry, his classification of the representations of the Poincaré group by mass and spin) to mention just some of the principal actors.

The enthusiasm for the promising role of group theory for (quantum) physics by some brilliant (mathematical) physicists, and the text books by Weyl [18], Wigner [19] and Van der Waerden [17] irritated or frightened others, who spoke of the *Gruppenpest*. For them group theory was just a new fancy way of deriving results that could already be understood by other older means.

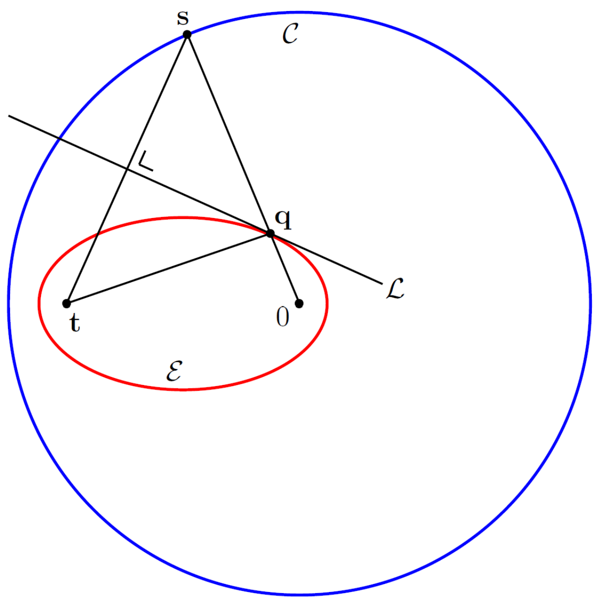

Figure 2 *Hidden symmetries in the Kepler problem*

In the physics community, Noether's principle reappeared around 1950 in the Lagrangian formulation of quantum field theory. The awakening of Noether's principle in the mathematics community started around 1970 with the revival of symplectic geometry, as discussed in the sections before.

In symplectic geometry, the equivalence of conservation law and symmetry amounts to the equivalence of function and corresponding Hamilton field. Hence the group law on the conservation laws is independent of the given Hamiltonian in contrast with the Lagrangian setting. It gave rise to the concept of moment map without a given Hamiltonian, with its multitude of applications. In recent times several books have been written about Emmy Noether [6], [15], [14] with the focus mainly on Noether's principle.

Hidden Symmetries in the Kepler problem

For visible symmetries, that is symmetries of a configuration space Q with induced actions on cotangent and tangent bundle, the link with conservation laws in the Hamiltonian or in the Lagrangian setting is clear and the two view points are easily related. However, for conservation laws that do not fit this scheme it can require some further thinking to make the underlying hidden symmetries visible. We will discuss this for the Kepler problem of planetary motion in the Hamiltonian picture.

Newton's equation of motion boils down in the Hamiltonian form to

$$[\dot{\mathbf{q}} = \mathbf{p}, \ \dot{\mathbf{p}} = -\mathbf{q}/q^3]$$

in suitable units, and so the Hamiltonian $H(\mathbf{q},\mathbf{p}) = p^2/2 - 1/q$ is conserved.

Besides the angular momentum vector $\mathbf{L} = \mathbf{q} \times \mathbf{p}$, whose conservation comes from central rotational symmetry, there is also the conserved Lenz vector

$$\mathbf{K} = \mathbf{p} \times \mathbf{L} - \mathbf{q}/q = (p^2 - 1/q)\mathbf{q} - (\mathbf{q} \cdot \mathbf{p})\mathbf{p}$$

whose hidden symmetry at first sight is unclear. Conservation of $\mathbf{L}$ implies that the motion is planar and satisfies the area law. For a fixed negative value $\mathrm{H}<0$ the motion is bounded inside the the disc $\mathrm{q} \leq -1/\mathrm{H}$. The boundary $\mathcal{C}$ is called the fall circle, because starting from there points with zero speed fall along a straight line onto the origin with infinite speed at the collision. The conservation of $\mathbf{K}$ follows by a direct calculation but writing it down seems at first black magic.

Let $\mathbf{s}=-\mathbf{q}/(\mathrm{qH})$ be the central projection on $\mathcal{C}$ of a point $\mathbf{q}$ inside $\mathcal{C}$ and let $\mathcal{L}$ be the tangent line to the orbit $\mathcal{E}$ with initial data $(\mathbf{q},\mathbf{p})$. By a direct calculation one can check that the reflection of $\mathbf{s}$ with mirror $\mathcal{L}$ is equal to $\mathbf{t}=\mathbf{K}/\mathrm{H}$ and so remains conserved. Hence the orbit $\mathcal{E}$ is an ellipse with long axis $-\frac{1}{\mathrm{H}}$. This quick geometric proof was found while teaching a masterclass for high school students on the Kepler laws [3].

The Lenz vector has been introduced (and forgotten) many times before its rediscovery by Lenz in 1924, but became well known after Pauli's use of its quantized version for the determination of the Hydrogen spectrum. The essence of Pauli's argument was the calculation of the commutation relations of $\mathbf{L}$ and $\mathbf{K}$ [13]. The associated Poisson brackets of $\mathbf{L}$ and $\mathbf{K}$ are given by

$$\{\mathrm{L_i},\mathrm{L_j}\}=-\varepsilon_{\mathrm{ijk}}\mathrm{L_k},\ \{\mathrm{L_i},\mathrm{K_j}\}=-\varepsilon_{\mathrm{ijk}}\mathrm{K_k},\ \{\mathrm{K_i},\mathrm{K_j}\}=2\mathrm{H}\varepsilon_{\mathrm{ijk}}\mathrm{L_k}$$

with $\varepsilon_{\mathrm{ijk}}$ the Levi-Civita symbol, which on that part of phase space with $\mathrm{H}(\mathbf{q},\mathbf{p})<0$ – so with bounded motion – amount to the symmetry of the Lie algebra $\mathfrak{so}_4(\mathbb{R})$.

In the Hamiltonian picture this symmetry was beautifully explained in geometric terms by Jürgen Moser in 1970 [8], [5]. In the Lagrangian picture Noether's converse theorem, that all conservation laws come from symmetry, is discussed for the Lenz vector in Section 5.3 of Olver's book [11]. One needs a formal variational calculus to explain this hidden symmetry, but the geometry remains somewhat obscure.

The Emmy Noetherweg

During my last visit to Leiden University this fall I noticed that there is now the Emmy Noetherweg (`weg' is Dutch for `road'). It is a good idea to honour such a distinguished mathematician. All nearby streets in the science campus are named after famous physicists

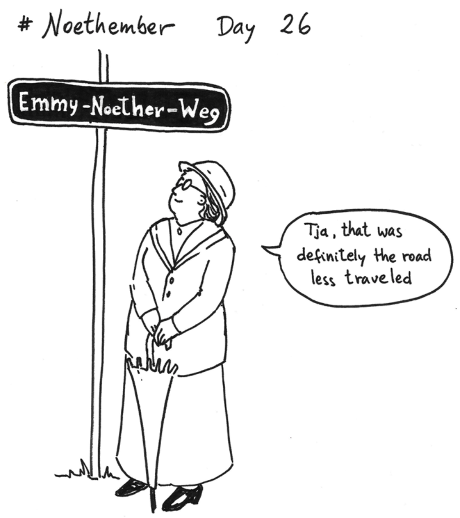

Figure 3 *The road in Leiden named after Emmy Noether*

Illustration: Constanza Rojas-Molina

like Paul Ehrenfest, Albert Einstein and Niels Bohr. So I presume that this honour to her is primarily because of her principle on symmetry and conservation laws.

However, pure mathematicians will remember Noether above all for her influential work in abstract axiomatic algebra from the 20's on until her untimely death in 1935. In Göttingen she became the center of research in algebra with a group of young students around her (Max Deuring, Ernst Witt, ...). The influential text book *Moderne Algebra* by Van der Waerden [16] – based on lectures by Emil Artin and Emmy Noether – helped spread the gospel. The abstract algebra of Hilbert and Noether was the key for the subsequent rigorous foundation of algebraic geometry by Van der Waerden, Weil and Zariski.

Today, Emmy Noether is an icon not just for her mathematics, but also for the discrimination she had to endure during her mathematical life for being a woman, first at Erlangen and from 1916 on at Göttingen, and then in 1933 also for being Jewish. Emmy Noether shared this same fate of twofold discrimination with Lise Meitner, an icon for physics. A block away from the Emmy Noetherweg there is also the Lise Meitnerweg.

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