Nelly Litvak
*Department of Mathematics and Computer Science*
*Eindhoven University of Technology*
*n.v.litvak@tue.nl*

Fulya Kula
*Faculty of EEMCS*
*University of Twente*
*f.kula@utwente.nl*

**Column** Better than blackboard

# The prior knowledge problem

**Welcome to the new issue of this column, with cheerful illustrations by Mara Chelărescu, a third year Applied Mathematics student at Eindhoven University of Technology. Today's column is co-authored with Fulya Kula, assistant professor at the Applied Mathematics department at the University of Twente; Fulya teachers mathematics and does research on mathematics education. Today we will discuss the problem that is very close to our heart: the highly varying and often lacking level of prior knowledge of the students. We know that many teachers will recognize this problem, and we hope to offer you some practical insights and workable solutions.**

Since we both teach probability and statistics, a first university course in probability is a typical example that comes to our mind. Such course usually requires some modest background in calculus: very standard series, derivatives, integration, and ... actually, not much more than that. Normally, we shouldn't expect any problem, all these are very basic prior knowledge. Yet, if you ever taught a first probability course, you know that this simple calculus can be a true struggle when students bump into it. Suddenly, they cannot sum up geometric series, don't recognize integration by parts, let alone the hopeless affair of swapping the order of integration or summation. Even errors in adding fractions are not unheard of; not once we personally graded exams with $\frac{1}{2}+\frac{1}{3}=\frac{2}{5}$ in them. What is it? Have the students learned nothing at high school and in their previous courses?

Of course, not all students struggle with prior knowledge. But many do, and then we, as teacher, are left with yet another problem: catering to the large differences in the students' levels.

What can we do about it? Before we get to the action plan, let's look at the problem in more detail.

**The problems with prior knowledge**
Based on the experience only, we may identify four different aspects of the prior knowledge problem.

1. Students forgot what they learned before.
2. Students don't recognize familiar knowledge in a new context.
3. Students don't know which specific knowledge from pre-requisite courses will be needed.
4. There are large differences in students' levels.

Let us look into these aspects in more detail one by one.

**Students forget what they learned before.**
Of course, students do forget what they learned in their previous courses. This is all but surprising.

First of all, being able to forget, is a fundamental feature of human memory. Think about your own university courses on topics that you don't encounter anymore. Most likely, you forgot large part of it, even if you were a diligent student, and knew it well at some point.

Students of course saw the material more recently. Then, those who studied well, will probably have the required prior knowledge. However, oftentimes, they didn't learn this material well to begin with. In the last column [3] we already discussed the necessary conditions for a human brain to learn: engagement, inconsequential error feedback, and spacing. Engagement means that the students actively work during the class, rather than passively listening to the teacher. Inconsequential error feedback means that students have safe space to make mistakes and hear how to improve, without any negative consequences such as grade reduction. And spacing means that students learn in small portions at regular intervals.

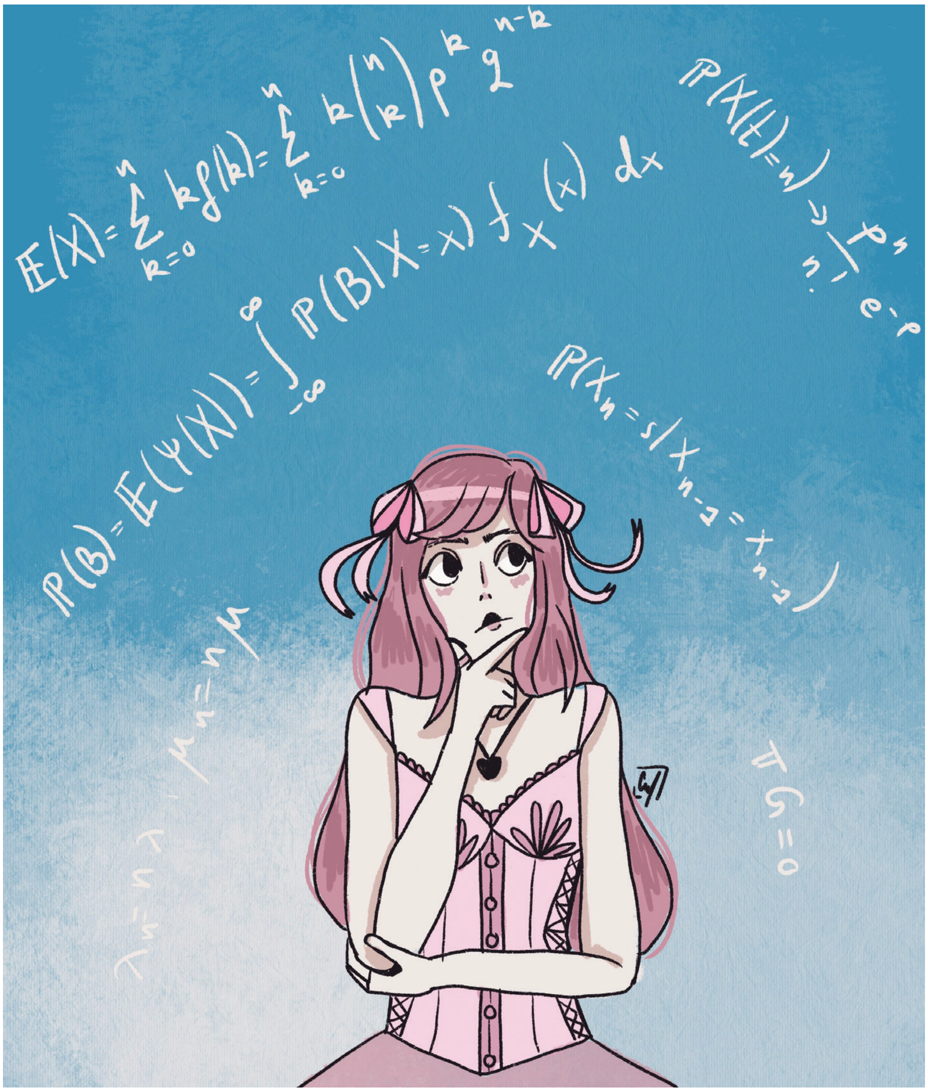
We have already discussed in [3] that the classical lecture-tutorial-exam course design with non-mandatory attendance, leaves it up to the students whether or not to follow these learning strategies. What we call 'treating students as adults' and 'giving them freedom to learn in their own way' de facto becomes the freedom to procrastinate, and to learn the entire course in the last week before the test. We cannot blame the students. Many people in the professional world work towards the deadline. We also cannot say that this last-minute learning is not successful because for most students, success means passing the test, and one can pass the test this way. The problem is that our human brain cannot digest 5 ECTS worth of information in one week, so the students forget great deal of it after the exam. Indeed, research shows [5] that practicing mathematics consistently over time increases retention, while excessive repetition does not. And by the way, if your course doesn't include some kind of regular, mandatory, grade-free or low-stake work with some kind of feedback, be assured that students will forget most of what they learned in your course soon enough. There are many ways to organize such mandatory work with feedback, see for instance our previous column [4]. But today we will discuss how we, as teachers, can we help students to recall the old material without us explaining it all over again.

### Familiar knowledge in a new context

One day very long ago due to a chain of random events, Nelly happened to take part in a tv commercial for plastic debit cards, not a common phenomenon at that time. The commercial was shown every day on a local television. At the same time, Nelly gave tutorials in probability at university. She expected that students will ask her about the commercial, but they didn't. In fact, nobody recognized her at all! This is a common phenomenon. If we know a person in one context, we might not recognize them in a different context.

Same is true for mathematics. In the analysis course, we did integration by parts, while in the probability course, we compute expectations. For the teacher, this is obviously the same thing. But the students might not recognize familiar material in a different context, with different notations, different intuition and interpretation.

Conclusion? Maybe we should explicitly place the old material in a new context.




Students don't recognize the hold material in the new context and don't know which of the pre-knowledge will be needed in the new course.

Illustration: Mara Chelărescu

### Which specific prior knowledge is needed?

Usually in the course description we write which courses are pre-requisites for our course. But we usually don't need the entire pre-requisite material, only some specific parts of it. Now let's look at it from the students' perspective. Suppose they passed the pre-requisite course with a seven out of ten. Does it mean that they have mastered 70% of the topics? What if they skipped exactly over the topics relevant for my course? Or maybe they know 70% of each topic? This is a problem, too, because when we use the old material we assume a complete fluency!

Altogether, from students' perspective, each course is a massive amount of information. We can visualize it as, say, a heap of facts, statements and formulas. For a student who passed with 6 or 7, this heap is not really structured. So it is hard for them to fish out a definition or an equation without a clear direction where to look.

Can we, as teachers, give them such direction? Yes, we can, and we probably should.

### Large differences in students' levels

The difference in level is especially visible when students have to use their prior knowledge. Some require explanation from scratch, others get bored and complain in evaluations that teacher spent too much time on what they already knew before. This is perhaps the most common problem that teachers experience with prior knowledge of their class.

One may say, there are strong students and weak students. This is probably a fact of life, but stating this it is not very helpful because we have to teach all students anyway. Plus, it is fair to say that our tendency to categorize very young individuals into strong and weak based on their exam results, is very close to the fallacy of a fixed mindset. Maybe our not-so-strong students are just not-so-strong yet? And if they are willing to try, it is our responsibility, as teachers, to give them a fair chance, over and over again. This is exactly what teachers are for, aren't they?

**What we did**

Both of us encountered the situations described above many times. And each of us had the case, when we addressed the prior knowledge problem heads on. No surprise, our solution is not in giving students more explanation. Instead, we suggest to let students do something that can help them to close their knowledge gaps. We don't say these are perfect solutions, but we found them reasonably effective. So let us tell you what each of us did.

**Homework quiz**

In 2022 Nelly was invited to give the Vacation Course (Vacantiecursus) organized yearly by the Dutch Platform for Mathematics (PWN). The audience are high school teachers of mathematics, and Nelly was going to teach them how to model the properties a real-life complex networks using random graphs. If you are interested, you may see an article based on this course, traditionally published in this journal [2].

Admittedly, despite 20+ years of teaching at university, Nelly found herself, for the first time, seriously thinking about the prior knowledge problem. Her course required good command of some elementary topics in probability: Bernoulli random variables; binomial distribution; Poisson distribution; expectation; linearity of expectations. Nelly had no idea how much of these her audience may know or remember. Most likely, some will know it all from on top of their head, while others never used probability and forgot it completely. So, what could she do?

The idea of giving a preliminary probability lecture didn't appeal to Nelly at all. The audience expects interesting modern mathematics, not a basic probability. Those who already know probability will be bored, and those who don't know probability, will not learn well enough from one lecture in order to appreciate random graphs.

So, instead of a preliminary lecture, Nelly decided to give a preliminary online homework. She thought, high school teachers should respect a homework, shouldn't they?

Nelly made a quiz using the Wooclap software and asked the organizers to distribute the link among the participants. The quiz had only six questions. After each question, there was the correct answer and a short video, that Nelly found on YouTube, that explained the material relevant to the question. If a participant struggled with the question, they could watch the video to refresh their knowledge.

**Quiz question:** Suppose you send $5$ packages. Packet $i$ arrives on time with probability $p_i$. Let $X_i$ be $1$ if packet $i$ has arrived on time, and $0$ otherwise, $i=1,2,3,4,5$. Let $Y$ be the total number of packages, out of $5$, that have arrived on time. We can write $Y$ as

$$Y = X_1 + X_2 + X_3 + X_4 + X_5.$$

Recall that $E(X_i) = p_i$, $i = 1,2,3,4,5$. Your friend states that $E(Y) = p_1 + p_2 + p_3 + p_4 + p_5$. Is this statement true or false? Choose the correct answer below.

A. The statement is always true.
B. The statement is false.
C. The statement is true if $X_i$'s are independent. Otherwise, it doesn't need to be true.

**Figure 1** Pre-knowledge quiz question: linearity of expectations.

Most difficult question turned out to be the one about the linearity of expectations; this question is shown in Figure 1. About 50% of participants chose option C. One participant wrote: "I was thinking too much and therefore made a mistake in $E(X+Y)$." However, this is a useful mistake. It makes one realize that the linearity of expectations is a deep and even somewhat counterintuitive property. (In fact, some probabilists even name linearity of expectations among most surprising facts in mathematics.) During the course, Nelly used linearity of expectations many times, it was crucial for understanding basic mathematical properties of random graphs. And from the reaction of the audience she could see that they recognized this result and understood the arguments.

At the end of the quiz, Nelly added two questions (#7 and #8) about the quiz itself. Question #7 was about the difficulty level. Nelly was pleased with the result: a nice normal distribution with 63% answering that the level was exactly right (see Figure 2). Clearly, the difference in opinions demonstrates the differences in prior knowledge. The quiz helped to level off these differences at least to the extent that made the course sufficiently accessible to everyone.

In Question #8, Nelly asked the participants to leave any other comments on the quiz. By far most common answer was: "Good refresher." Another frequent answer was: "I hope that the level of the course will be higher." Of course, the quiz was much easier than the course! However, it was also good that the quiz was doable because it did serve the goal of refreshing prior knowledge, and at the same time wasn't frustrating for the participants, and left them in anticipation of a bigger challenge.

Not everyone was equally enthusiastic. One participant wrote: "After making this quiz, I deregistered for the course." We don't know why. But Nelly hopes that maybe this person realized that they were not interested in the topic. And then deregistering is also a positive result. This person might have better things to do with their time, while those who chose to attend were a wonderful audience: curious, motivated, and well prepared.


**Figure 2** Answers of the course participants about the difficulty level.

## Bridging course

Across the years, Fulya consistently found herself explaining high-school-level content when teaching mathematics to university students in engineering programs. This situation persisted and even worsened over time. As a teacher, Fulya faced a dilemma: should she prioritize helping the mathematically weak students, potentially ignoring the mathematically strong ones, or should she adhere strictly to the curriculum and risk leaving struggling students behind? This dilemma is undesirable for any teacher. Seeking an effective solution, Fulya created an online Bridging Course partially funded by the 4TU.Center for Engineering Education.

The Bridging Course, delivered entirely online via the Canvas learning management system, aims to assist students with high school mathematics subjects. The course was initially developed to support the first Calculus course for engineering majors. After much thinking and discussions, Fulya decided to exclude the high-school background in limits, derivatives, and integration because these topics are covered extensively at the university. Since students have only very limited time to refresh their prior knowledge, this time is better spent on more basic but fundamental high-school mathematics.

The Bridging Course has three main components: numbers, functions, and trigonometry. Within each of these parts, there are specific subtopics, and each subtopic has clearly defined learning objectives. Each learning objective starts with a related question.

*If the student answers this question correctly*, they have the option to proceed to the next learning objective.

*If the answer is incorrect*, the student is directed to a brief instructive video that covers the topic. The videos are intentionally concise, usually around 4–5 minutes, to cater to the famous short attention spans of today's young adults. Some of the videos were recorded by Fulya's fantastic student assistants, Lavinia Lanting and Linda ten Klooster. (You already met Lavinia in our previous column [4]). Fulya checked their slides, and then Linda and Lavinia recorded the videos in the DIY studio of the University of Twente. This saved Fulya lots of time, and it was nice for the students to see young faces in the videos. Other videos Fulya found online, e.g. at Khan's Academy. All videos are interactive, meaning that while or after watching the video, students get follow-up questions that require short answers, reinforcing their understanding of the topic. The questions were inserted with H5P software. In Figure 3 we show an example of such follow-up (drag-and drop) question.

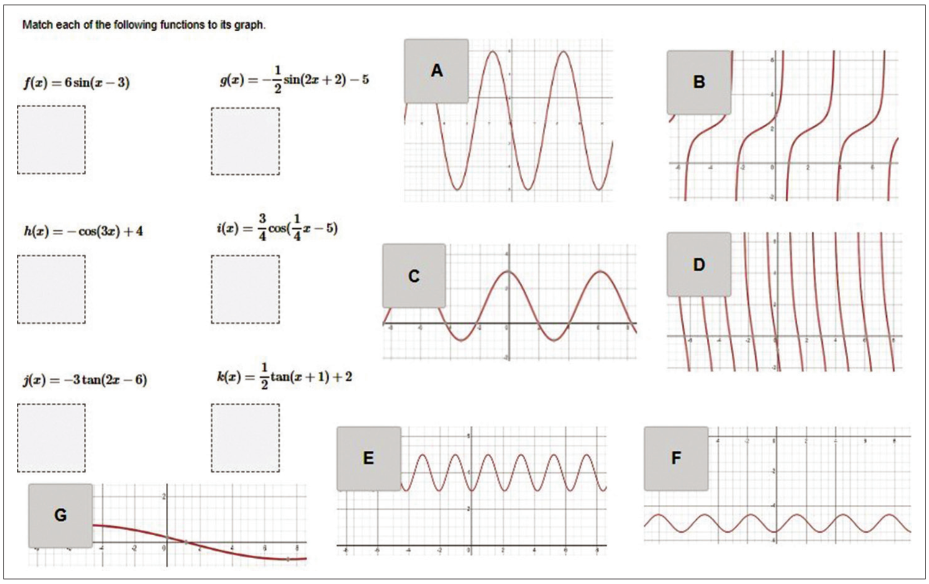



**Figure 3** An example of a drag-and-drop question in a bridging course.

For self-recorded videos, inserting questions requires some time but technically is easy to do. For the video's found online, some more technical steps are needed. This part was done by Heleen van der Zaag of the Technology Enhanced Learning and Teaching (TELT) team at the University of Twente. This is a perfect example where

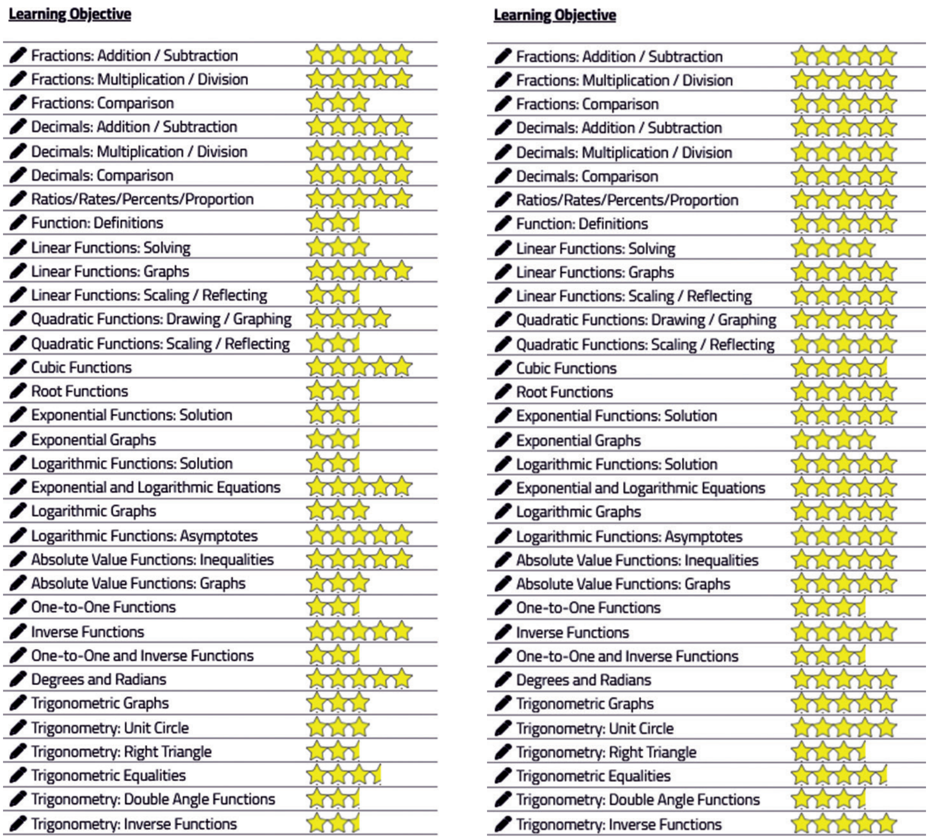



**Figure 4** Two examples of student feedback in the Bridging Course.

teaching support team was there for the teacher and her students! After completing the instructive video and answering the follow-up questions, students move on to solve a problem similar to the initial question posed at the beginning of the learning objective, that the student failed to answer correctly.

This structured approach is repeated for each learning objective in the Bridging Course.

Moreover, an essential part of the Bridging Course is learning analytics. Ethically approved and with a student's consent, feedback is provided to both students and teachers based on collected data.

*Student feedback* is given through a star-based system where even minimal effort is acknowledged. If a student earns minimum 2.5 stars or ideally more out of 5, they see that they have been mastering the subject. If they receive fewer than 2.5 stars, they understand that they need to dedicate more effort. Figure 4 shows two examples of such student feedback. The student on the left clearly has more struggles with prior knowledge compared to the student on the right; this student (on the left) can also see which subjects need additional work.

*Teacher feedback* is provided through heatmap visualizations for the overall class overview. Figure 5 shows an example of teacher feedback. The legend in Figure 5 explains the colors: the green color means that the student has answered the question correctly; the yellow color means that the student didn't answer correctly; the darker the color, the more attempts the student made.

In case of multiple attempts, the students attempt the same question. As you see from the example in Figure 3, it is highly unlikely to merely guess the answer after a few random attempts; and if a student made many attempts, then the dark color will fairly label the question as a difficult one. Importantly, the students have no incentive to cheat because there is no grade attached to the Bridging Course. The course is there to help the student, and has no other goal.

The teacher can easily see which topics are difficult by looking at the columns, one column per topic. Darker columns mean more attempts, so these are most difficult topics. In Figure 5, the students seem to struggle with inverse functions: see the columns for 'Root functions', 'Inverse functions', '1-1 and Inverse', 'Trig: Inverse functions'. If desired, the teacher can also look at the rows for student-specific data.

The course was for the first time implemented in 2023 at the Civil Engineering program of the University of Twente. Both students and teachers greatly appreciated the course. Students valued the provided feedback. Moreover, they emphasized that it was helpful to understand their readiness for the first calculus course. They also appreciated that it was clearly indicated which subtopics required further studies; students said they now knew what was expected of them.

Feedback from teachers indicated that the course significantly saved their time by providing a clear overview of the class. They now had the option to provide supplementary materials or clarify specific topics as needed.

Recognizing the widespread need across different programs to bridge gaps in preliminary knowledge of mathematics, the Bridging Course is being prepared for implementation in several programs at the University of Twente starting September 2024.

The Bridging Course is openly available for use in its current form and as a framework for preliminary catch-up in any large-scale course. The shell is ready; the teachers will need to populate it with questions and videos. While this may initially be time-consuming, it will be a one-time effort that can be used consistently over the years. If you are interested, feel free to visit: https://www.utwente.nl/bridge.


**Figure 5** Heatmap Visualization of students' performance in the Bridging Course. Darker colors (green or yellow) show larger number of attempts; green means that eventually the student answered correctly; yellow color means that the student didn't answer correctly.

**Solving the prior knowledge problem**

Now you know what solution we offer to the prior knowledge problem: Give your students the means to check and refresh their understanding of the specific prior knowledge relevant for your course. Best to do it online, in a structural way, keep the questions simple, and stick to the prior knowledge you truly need.

If your course is large, you may use Fulya's Bridging Course shell and populate it with questions. But even a small quiz like Nelly made for high-school teachers will do a good job.

With this approach, you can address all four aspects of the prior knowledge problem, that we identified before. Let's look at them again.

*1. Students forgot what they learned before.*
Based on the feedback of the learners, Quiz or Bridging Course are a good refresher.

*2. Students don't recognize familiar knowledge in a new context.*
In a Quiz or Bridging course you may already use the notations and formulations needed in your course. This will directly tie the prior knowledge to the context of your course.

*3. Students don't know which specific knowledge from pre-requisite courses will be needed.*
This is a great strength of the Quiz or Bridging Course. You can choose exactly the relevant questions. For the students, this will be much more specific and actionable than the list of pre-requisite courses.

Indeed, suppose that instead of giving a Quiz, Nelly wrote to her course participants that they would need some knowledge of probability. First of all, this is too large for them to review, so most likely all of them would come unprepared. But even if some participants decide to review something, how will they know what exactly to review? For instance, they may choose to review the Normal distribution, that is central to basic probability courses, but it is not very relevant for random graphs. Now imagine the mutual frustration: Nelly struggles because the participants don't remember the Binomial and the Poisson distributions; participants, in their turn, are disappointed because they did their best to review probability, and this was as good as useless!

Illustration: Mara Chelărescu

The bridge. A bridging course helps the student to cross over from the high school to the university.

Similarly, in the Fulya's Bridging course, it was crucial that the questions targeted exactly the knowledge needed for the first-year calculus course. This way, the students received very specific directions what to review, instead of the general pre-requisite such as 'at least a 6 for high-school Mathematics B'.

*4. There are large differences in students' levels.*
The Quiz or Bridging course give all students a chance to catch up and succeed in a course. As teacher, this releases you from the eternal catch-22 situation where you either explain the old material, boring for the strong students, or proceed with the curriculum, losing the rest of the class. The prior knowledge Quiz or Bridging Course gives you the third option: remind the students which Quiz question is relevant for this specific topic in your course. Even if they don't remember, they will know where to look, and how to get up to speed.

**Rippling effects**

We believe that our approach, driven by empathy to the students' natural struggles, may have significant rippling effects. The students will appreciate our understanding and support. Reviewing old material will cement their university knowledge. The benefits may go even further. For instance, there are good reasons to hope that Fulya's Bridging Course will reduce dropout. Indeed, studies on the reasons behind dropout suggest that mathematics teaching should include support for students to overcome challenges, and should help them to reflect on their learning experiences [1]. Support with challenges and reflection on their learning are exactly what Bridging Course offers to the students.

**It doesn't have to be a lot of work**

Is it a lot of work? It depends. The Bridging Course as Fulya did it, is a substantial effort. It is worth doing for large courses, and you will need a lot of time, reliable student assistants, and good support.

But the Quiz is quite light. Once Nelly discussed it with a colleague, and they said that for their course, 5-6 questions would definitely be enough, just like Nelly did it in her Quiz for the high school teachers. If you think about it, you will see that for your course, you don't need many more either. Setting a Quiz of 5-6 questions in Canvas or any other learning environment, is not hard at all.

The videos are very useful, all our learners are unanimous about it. But the good news is that the Internet is full of really nice videos on all topics. Recording a video is a lot of work, but it is not very hard to find a video that will be sufficiently helpful.

We know that everyone is very busy. But if you have a chance, we hope you will see this column as call for action. If your course suffers from the prior knowledge problem, give your students a refresher. Complaining about students' prior knowledge will not take us further. Giving them means to catch up — just might.

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