Problemen

| Problem Section

This Problem Section is open to everyone; everybody is encouraged to send in solutions and propose problems. Group contributions are welcome. We will select the most elegant solutions for publication. For this, solutions should be received before **15 April 2024**. The solutions of the problems in this issue will appear in one of the subsequent issues.

Problem A

Prove that for all three line segments of length 1 in a closed disc of radius 1 there are two of those line segments with distance at most 1.

Problem B (proposed by Hendrik Lenstra)

Let A be an abelian group and write $\mathrm{End}(A)$ for the ring of group homomorphisms from A to A. Show that A is free as $\mathrm{End}(A)$-module if and only if A admits a commutative ring structure so that the Cayley map $A \to \mathrm{End}(A)$ given by $x \mapsto (y \mapsto xy)$ is an A-module isomorphism. Show that for all subrings of $\mathbb{Q}$ and rings $\mathbb{Z}/p\mathbb{Z}$ with p a prime the Cayley map is an isomorphism, and give an example of an uncountable ring with this property.

Problem C (proposed by Albert Visser)

A *Gollum ring* is a ring that is isomorphic to the subring $\mathbb{Z}+2R$ of a commutative ring R. Show that there is a sentence in first-order logic in the language of rings that is true in all Gollum rings, but not true in all commutative rings.

Edition 2023-3 We received solutions from Rik Biel, Chris A.J. Klaassen, Thijmen Krebs, Lucía L. Pacios, Ana Pose, Andrés Ventas and Jan de Vries.

Problem 2023-3/A

Let $n>0$ be an integer and let $\varphi:\mathbb{R}^n \to \mathbb{R}^n$ be an isometry, i.e., a map such that for all $x,y\in\mathbb{R}^n$ we have $|\varphi(x)-\varphi(y)|=|x-y|$. Let $X\subset\mathbb{R}^n$ be a set such that $\{\varphi(x)\mid x\in X\}\subseteq X$. Show that if X is closed and bounded, then $\{\varphi(x)\mid x\in X\}=X$, and show that we can drop neither of these two assumptions.

Solution This problem was solved in a collaboration by Lucía L. Pacios and Ana Pose and Andrés Ventas. Moreover, it was solved by Rik Biel and Thijmen Krebs. Jan de Vries has proved a more general result: an isometry on a compact metric space is surjective.

Let $y\in X$. By continuity of φ it suffices to prove that for every $\varepsilon>0$ there exists $x\in X$ such that $|\varphi(x)-y|<\varepsilon$. By Bolzano–Weierstrass, there exist $i>j$ such that $\left|\varphi^i(y)-\varphi^j(y)\right|<\varepsilon$. Using that φ is an isometry, we then find that $\left|\varphi^{i-j}(y)-y\right|<\varepsilon$. Now $x=\varphi^{i-j-1}(y)$ suffices.

Consider $X=\mathbb{Z}_{\geq 0}\subseteq\mathbb{R}$. Note that X is closed but not bounded, and that $x\mapsto x+1$ is an isometry that maps X to $X\backslash\{0\}$.

Identify $\mathbb{R}^2$ with $\mathbb{C}$ and take $X=\{\exp(n\mathrm{i})\mid n\in\mathbb{Z}_{\geq 0}\}$. Note that X is bounded but not closed, and that $x\mapsto\exp(\mathrm{i})\cdot x$ is an isometry that maps X to $X\backslash\{1\}$.

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Problem 2023-3/B

Let X be a normally distributed random variable and let $t\in\mathbb{R}_{>0}$. Show that $x\mapsto\mathbb{P}(X\leq x+t\mid x\leq X)$ is an increasing function.

Oplossingen | Solutions

Solution This problem contained an error. Instead of ‘increasing’, it was stated ‘decreasing’. A solution for the problem was sent in by Thijmen Krebs. A more general result was proven by Chris A.J. Klaassen: instead of for normal distributed random variables, it has been proved for random variables with a strongly unimodel distribution.

First note that without loss of generality we may assume that X follows a standard normal distribution, and write f and F for its probability density and cumulative density function, respectively. It suffices to show that $G(x) = F(x+t)/F(x)$ is a decreasing function, or equivalently, that

$$(\log G(x))' = \frac{f(x+t)F(x) - f(x)F(x+t)}{F(x)F(x+t)} < 0$$

for all x. Hence it suffices to show for all x that

$$H(x) = e^{-tx-\frac{1}{2}t^2}F(x) - F(x+t) < 0.$$

We claim that

$$\lim_{x\to-\infty} e^{-tx}F(x) = 0.$$

Clearly 0 is a lower bound. Let $\delta > 0$. Then there exists some $B < 0$ such that $e^{-\frac{1}{2}x^2} \le \delta e^{tx}$ for all $x < B$. Hence

$$F(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}z^2}\mathrm{d}z \le \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \delta e^{tz}\mathrm{d}z = \frac{\delta}{\sqrt{2\pi}}\cdot e^{tx}$$

for such x, so $e^{-tx}F(x) \le \frac{\delta}{\sqrt{2\pi}}$ for x sufficiently small. Taking $\delta \to 0$ we obtain the limit.

It follows from the claim that $\lim_{x\to-\infty} H(x) = 0$. Since also

$$H'(x) = -te^{-tx-\frac{1}{2}t^2}F(x) + e^{-tx-\frac{1}{2}t^2}f(x) - f(x+t) = -te^{-tx-\frac{1}{2}t^2}F(x) < 0,$$

we conclude that $H(x) < 0$ for all x.

Problem 2023-3/C (proposed by Hendrik Lenstra)

Let $p = 2n+1$ be an odd prime and consider the finitely presented group G with generators $x_1,\dots,x_n$ and for each $0 < i,j,k \le n$ such that $ij = k$ or $ij = p-k$ the relation $x_ix_j = x_k$. Show that G is a cyclic group of order n.

Solution A partial solution for this problem was given by Andrés Ventas.

We prove by induction on i the following statement:

$$\forall j,k \in \{1,\dots,n\},\ \ (ij \equiv \pm k \bmod p \ \Rightarrow\ x_ix_j = x_jx_i = x_k).$$

This is trivial for $i = 1$. Now fix an integer $1 < i \le n$ and assume that the statement holds for smaller values of i. We prove the statement for i. Let $j,k \in \{1,\dots,n\}$ be such that $ij \equiv \pm k \bmod p$. Consider the i sets $\{hj, hj+1, \dots, hj+\lfloor p/i\rfloor\}$ for $h = 0,\dots,i-1$. Each of these sets contains $\lfloor p/i\rfloor + 1$ elements, and as $i\cdot(\lfloor p/i\rfloor+1) > p$, we find that the sets are not pairwise disjoint modulo p. It follows that there exist integers $0 \le h_1 < h_2 < i$ and an integer $0 \le \varepsilon < p/i$ for which $h_2 j \equiv h_1 j \pm \varepsilon \bmod p$, and $h = h_2 - h_1$ then yields $hj \equiv \pm\varepsilon \bmod p$. Note that $\varepsilon \ne 0$ since p is prime. By assumption we have $x_ix_\varepsilon = x_{i\varepsilon}$ and by the induction hypothesis applied to h we know that $x_hx_i = x_{ih} = x_ix_h$ and that $x_hx_k = x_{i\varepsilon} = x_kx_h$. This gives the following equalities

$$x_ix_jx_h = x_ix_\varepsilon = x_{i\varepsilon} = x_kx_h,$$
$$x_hx_jx_i = x_\varepsilon x_i = x_{\varepsilon i} = x_hx_k,$$

and thus $x_ix_j = x_k = x_jx_i$ which completes the induction step. Now it follows that $G \to \mathbf{F}_p^*/\{\pm 1\}$ given by $x_i \mapsto i$ for all $i \in \{1,\dots,n\}$ is an isomorphism. Since $\mathbf{F}_p^*$ is cyclic of order $2n$, it follows that G is cyclic of order n.