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Column New Impulse

Game theory and two-way interactions between mathematics and economics

In this column, researchers who have been recently appointed to one of the Dutch mathematics institutes, introduce themselves.

I am a professor of Game Theory at the Mathematical Institute at Utrecht University. Prior to joining Utrecht, I have held tenure-track positions at Northwester University and at Oxford.

Game theory — or, interactive decision theory — analyzes the optimal behavior of decision-makers (*players*) whose decisions affect each other. Fittingly, the field's origins lie in the study of parlor games (e.g., chess, poker, bridge), with important papers by Ernst Zermelo (on chess) and Émile Borel (inspired by card games). Mathematically, the important feature of parlor games is that they are strictly competitive (i.e., zero sum): One player's gain is the other player's loss. A cornerstone of modern game theory is the minimax theorem proved by John von Neumann in 1928: for every two-player zero-sum game (that satisfies some conditions), there is a decision rule (*strategy*) for each player that minimizes the player's worst-case loss. Despite the continuing importance of this result, it does have significant limitations: Many games of interest have more than two players or have some element of cooperation. It took 22 years before mathematicians were able to analyze general games. The key insight is that the strategies in von Neumann's minimax solution do not only limit a player's worst-case loss, they are also mutually optimal. That is, a pair of minimax strategies is a fixed point of the correspondence that maps any strategy s_i for player i to strategies for player $j \neq i$ that maximize j's (expected) reward if i chooses s_i. This idea is entirely general, and in 1950, John Nash showed that every sufficiently well-behaved game has such a fixed point (equilibrium).

In the decades since Nash's work, game theory has expanded its scope tremendously, making connections to areas as diverse as probability theory and statistics [1], stochastic processes [16], dynamical systems [8], tropical geometry [3], algebraic topology [6], differential topology [9], general topology [14], axiomatic set theory [5], model theory [4], category theory [12], epistemic logic [2], computational complexity theory [13], order theory [15] and partial differential equations [10], to name but a few. Nevertheless, fixed points continue to play a central role throughout game theory. Indeed, somewhat tongue-in-cheek, one could say that game theory can be viewed as the study of fixed points. In my view, this is a source of strength: Game-theoretic results developed in one context regularly can be used to prove results in completely different contexts, sometimes outside of game theory proper.

Much of my own research centers around players' beliefs: What they think others may do. Also there fixed points show up in sometimes unexpected ways. One example is an ongoing project that studies when players' beliefs have a 'simple' representation. To illustrate, suppose there are two players, Ann and Bob. We say that Ann is *rational* if she chooses a strategy that maximizes her (expected) reward given her belief (a probability measure over Bob's strategies); likewise for Bob. Where do players' beliefs come from? Nash assumed that players' beliefs are correct: Ann puts probability 1 on Bob's actual strategy, and analogously for Bob. But in many cases this assumption is too strong. A weaker assumption is that players are rational, believe that the other is rational (i.e., assign probability 1 to the other player being rational), believe that the other believes that the other is rational, and so on. This assumption, commonly termed *rationality and common belief of rationality* (RCBR), requires considering an entire hierarchy of beliefs: probability measures on sets of probability measures on sets of probability measures,

and so on. Belief hierarchies are mathematically well-defined but can be difficult to work with in practice. To address this, John Harsanyi showed that belief hierarchies have a simple recursive description: Belief hierarchies can be modeled by endowing each player i with a set of 'types' T_i, where each type $t_a \in T_a$ for Ann is associated with a probability measure over Bob's strategies and Bob's types; again, analogously for Bob [7]. Then, each type t_a for Ann unwinds into an infinite hierarchy of beliefs that specifies her beliefs about Bob's strategy, about his beliefs about her strategy, and so on. While elegant, Harsanyi's approach opens up a new question: Does there exist a situation for which there is no set of types large enough to model all belief hierarchies that we are interested in? It turns out that, for the case sketched above, no such situation exists: types can generate all belief hierarchies (under some topological or measure-theoretic conditions); moreover, they can generate all predictions consistent with RCBR [11]. But, for some types of games, using probability measures to model beliefs is not appropriate. For example, in 'dynamic games' that unfold over time, we have to specify how players update their beliefs when they receive new information. While type spaces and belief hierarchies can easily be defined for these more general settings, it is unclear that the results described above still go through: A key step in the proof of these results is that the strategies that can be played under RCBR are the fixed point of a monotone function (i.e., an order-preserving function). Yet, monotonicity often fails for more general notions of beliefs. My earlier work has shown that this need not be problematic: At least in some nonmonotone cases, Harsanyi's approach is sufficiently expressive. My present work aims to characterize when this holds in general. This would also help unify existing results for the case where beliefs can be represented by probability measures.

I was recruited to Utrecht to help set up a new double BSc program in mathematics and economics (together with my colleague Kees Oosterlee). Under this program, students earn two BSc degrees in three years: one in mathematics, and one in economics. The first students started in September 2022, and a new cohort has started this fall. I am very excited about the program. Mathematics has long played a central role in addressing questions of central economic relevance, and today's and tomorrow's economic problems, such as increasing inequality, will likewise require the attention of well-trained mathematicians with a thorough understanding of economics. To give an example, a central question in the 1950s was whether communism-style central planning systems could ever outperform capitalist market-based systems. Of course, over time evidence emerged that existing central planning systems were failing. But, without formal models, the question remained whether greater economic success could not be achieved by some new kind of central planning system. This question was settled by mathematical economists who characterized exactly when capitalist price systems are superior. And, as a testament to the power of abstract reasoning, essentially the same ideas used to settle this question can be used to analyze completely different phenomena, a recent example being Google's ad auctions!

To equip students' with the necessary tools, the new double BSc program goes well beyond BSc programs in econometrics: Because students complete both degrees, they have the opportunity to delve into topics that traditionally receive less attention within an econometrics program. To give an example, economic inequality tends to go hand-in-hand with limited social mobility. There are several explanations for this, and therefore different policy interventions that could be effective if one would like to reduce inequality or increase mobility. Disentangling those requires both mathematical modeling (dynamical systems, stochastic processes, game theory) and a good understanding of the underlying economic drivers and institutions. And it's not just economics that can benefit from a closer relation with mathematics: the reverse is also true. While there is less of a tradition of a two-way interaction between the two fields than with physics and mathematics, there are many examples where advances in mathematics had as their origin a purely economic motivation. It is to this two-way interaction that I hope to contribute at Utrecht, in game theory and beyond.

References

1 I. Arieli, Y. Babichenko, F. Sandomirskiy and O. Tamuz, Feasible joint posterior beliefs, *Journal of Political Economy* 129(9) (2021), 2546–2594.

2 R.J. Aumann, Agreeing to disagree, *Ann. Statist.* 4(6) (1976), 1236–1239.

3 E. Baldwin and P. Klemperer, Understanding preferences: 'demand types', and the existence of equilibrium with indivisibilities, *Econometrica* 87(3) (2019), 867–932.

4 A. Brandenburger and H.J. Keisler, An impossibility theorem on beliefs in games, *Studia Logica* 84(6) (2006), 211–240.

5 J.H. Conway, *On Numbers and Games*, London Mathematical Society Monographs, Vol. 6, Academic Press, 1976.

6 D. Gale, The game of Hex and the Brouwer fixed-point theorem, *Amer. Math. Monthly* 86(10) (1979), 818–827.

7 J.C. Harsanyi, Games with incomplete information played by 'Bayesian' players, I–III, Part I. The basic model, *Management Science* 14(3) (1967), 159–182.

8 J. Hofbauer and K. Sigmund, Evolutionary game dynamics, *Bull. Amer. Math. Soc. (N.S.)* 40(4) (2003), 479–519.

9 E. Kohlberg and J.-F. Mertens, On the strategic stability of equilibria, *Econometrica* 54(5) (1986), 1003–1037.

10 P.-L. Lions and J.-M. Lasry, Large investor trading impacts on volatility, Ann. Inst. H. Poincaré C Anal. Non Linéaire 24(2) (2007), 311–323.

11 J.-F. Mertens and S. Zamir, Formulation of Bayesian analysis for games with incomplete information, *International Journal of Game Theory* 14 (1985), 1–29.

12 L.S. Moss and I.D. Viglizzo, Harsanyi type spaces and final coalgebras constructed from satisfied theories, *Proceedings of the Workshop on Coalgebraic Methods in Computer Science*, Electron. Notes Theor. Comput. Sci., Vol. 106, Elsevier, 2004, pp. 279–295.

13 C.H. Papadimitriou, The complexity of finding Nash equilibria, in *Algorithmic Game Theory*, Cambridge University Press, 2007, pp. 29–52.

14 R. Telgársky, Topological games: On the 50th anniversary of the Banach–Mazur game, *Rocky Mountain Journal of Mathematics* 17(2) (1987), 227–276.

15 D.M. Topkis, Equilibrium points in nonzero-sum n-person submodular games, *SIAM Journal on Control and Optimization* 17(6) (1979), 773–787.

16 N. Vieille, Stochastic games: recent results, *Handbook of Game Theory*, Elsevier, 2002, pp. 1833–1850.