Eye on Variety – The teacher’s work in developing mathematical whole-class discussions
Chris Kooloos

In May 2022, Chris Kooloos from Radboud University Nijmegen successfully defended his PhD thesis with the title Eye on Variety – The teacher’s work in developing mathematical whole-class discussions. Chris carried out his research under the supervision of prof. dr. Gert J. Heckman (Radboud University), prof. dr. Rainier H. Kaenders (University of Bonn) and dr. Helma W. Oolbekkink-Marchand (Radboud University & HAN University of Applied Sciences).

Since 2021, Chris has been working as ‘vakdidacticus’ mathematics at Radboud University. In this position, he develops and provides master courses that are especially interesting for educational master students.

Teaching students to think mathematically
In the last century much has changed regarding mathematics education, attention has shifted from teaching mathematics as a set of rules and procedures toward teaching for mathematical thinking. The mathematical content should be supportive in — as well as resulting from this general analytical way of thinking. In the end, mathematics is at first a human activity, and not only a collection of procedures in which things need to be done in a certain way.

Research into student-centered mathematics education has flourished and provided many insights, models, and tools regarding teaching for mathematical thinking. In his thesis, Chris considered one specific teacher practice with regard to developing students’ mathematical thinking: discourse-based teaching, or, in other words, teaching through whole-class discourse that builds on students’ various ways of thinking. Discourse-based teaching relies on teaching mathematics lessons by

1. deciding what mathematics students should learn;
2. thinking about problems or questions that, if students work on them and discuss their ideas, can lead the students to learning to deal with that particular mathematics;
3. encouraging students to work on the problems and questions;
4. orchestrating a whole-class discussion about students’ various ideas and solution methods.

During his PhD, Chris collaborated with a group of Dutch mathematics teachers and supported them in developing mathematics lessons based on whole-class discussions. They all had a common vision. Namely, to create opportunities for their students to experience variation in mathematics, create a learning environment in class where students can discuss their mathematical ideas freely, experience that mathematics can be done in different ways and is something they already do, and to realize that by discussing
their thinking they collectively learn new mathematics! A very novel
vision to strive for.

Discussing to foster understanding
For teachers, discourse-based teaching is a valuable practice to
gain insight into students’ thinking, to build the lesson on their
thinking, and to guide them toward learning important mathemat-
ical ideas. In discourse-based mathematics lessons, students can
also develop a corresponding disposition: that their thinking mat-
ters, and that through thinking and discussing their thoughts, they
can discover important mathematical ideas. However, developing,
and orchestrating classroom discourse about students’ different
solution methods is a complex task for mathematics teachers.
Work, time, creativity, patience, and persistence are needed.

What makes the orchestration of productive classroom discourse
particularly complex is finding a balance between building on stu-
dents’ ideas and guiding the students toward domain-specific
goals, such as ‘understanding’ of mathematical concepts like tan-
gent lines. Mathematical goals can only be achieved by building on
student thinking and by supporting them in making the necessary
steps they need toward understanding the topic. The teacher has
the responsibility to make decisions about the ideas students share
and to advance the mathematical learning of the whole group to-
ward the disciplinary mathematical ideas. For classroom discourse
to be productive, students sharing and discussing ideas is not suf-
ficient. Some students’ mathematical ideas are more advanced than
others, some explanations are generalizable, and some are not.
Here lies an essential yet challenging task for the teacher, namely
orchestrating classroom discourse such that the students are both
supported in making important mathematical connections, and are
also guided toward disciplinary ideas. At the same time the focus
should be maintained on students’ ideas and reasoning.

A large part of Chris’s research is devoted to professional devel-
opment and teacher learning. He collaborated closely with a team
of mathematics teachers who wanted to implement discourse-
based teaching in their classrooms. Teachers were stimulated to
develop and implement discourse-based teaching while staying close to their practice and having regular meetings to discuss and
evaluate the progress. So, the work is very close to high school
teaching practice. During this collaboration, Chris investigated how
teachers learned and applied discourse-based teaching in their classrooms, how they reacted to the students’ ideas, and how they
tried to make sense of the students’ mathematical learning.

In four lessons
Let us have a peak in one of these collaborations. During his re-
search, Chris collaborated closely with tenth-grade teacher Anna in
developing four discourse-based lessons in analytic geometry. The four lessons in analytic geometry consisted of students working
on a mathematical problem plus classroom discourse concerning
students’ different solution methods. During the first lesson, the
students were asked to think of the following problem:

Calculate the distance from point $P(6,1)$ to line $L: y = \frac{1}{3} + 4$.

The current presentation of analytic geometry in Dutch textbooks
is very procedural and often consists of step-by-step instructions.
For example, to calculate the distance from a point to a line, only
one solution method is given. Generally, students are accustomed
to memorizing and practicing such step-by-step procedures. During
these lessons, instead of providing students with a single proce-
dure, Anna presented them with open problems, and orchestrated
classroom discourse about students’ different solution meth-
ods. When students were asked to calculate the distance from a
point to a line, several possible solution methods were formulated
and discussed.

Moreover, by comparing various solution methods, students
were supported in making important mathematical connections
between different representations. Effective orchestration of class-
room discourse shifts students’ cognitive attention from problem
solutions and procedural rules to sense-making and the reasoning
that leads to a solution. In other words, instead of trying to get the
right answer, mathematical discourse is about trying to understand
and question each other’s ideas and reasoning, and collaboratively
deciding what is true based on logical argumentation.

From the analysis of the lesson observations, it is observed that
the number of students who contributed to classroom discourse
increased during the course of the four lessons. In the fourth les-
son, 18 out of 23 students contributed to the discourse. In the first
lesson, Anna talked twice as much as the students; in the second
lesson, the students talked twice as much as Anna; and in the third
and fourth lessons, the students talked slightly more than her.

A framework for teacher actions
During whole-class discourse, teachers should make many decisions
on the spot, they need to build upon students’ thinking and at the
same time keep an ‘eye on the mathematical horizon’. Chris con-
structed a theoretical framework to characterize the teacher’s ac-
tions in classroom discourse, to describe the change in the teacher’s
role in learning discourse-based mathematical teaching, and to
describe how the teachers tried to make sense of the mathematical
thinking of the students.

The teacher’s actions — utterances — in interaction during whole-
class discourse are crucial in supporting the students to express
their mathematical thinking, engage with each other’s thinking,
and make important mathematical connections. In a way, the
teacher’s actions are the gateway between everything the teacher
knows, thinks, wants, and feels on the one hand and the students
on the other hand. A teacher can choose to give a hint or to ask an-
other student to think of the next step in trying to solve a problem.
The concrete actions the teachers choose to coordinate whole-class
discourse affect the learning process of the whole class.

A theoretical framework for the teacher’s actions can give con-
crete characteristics of the attitude and work of teachers during
classroom discourse. In the framework Chris developed teacher
actions are divided into four categories. Those are: convergent actions (that help converge to a solution) such as demonstration and reformulation, divergent actions (that stimulate interaction and discussion between students) such as requesting explanations and clarifications, encouraging actions such as confirmation and encouragement, and regulatory actions, to make sure that the discussions are regulated. A combination of teacher actions can guarantee that there is a balance between building on students’ ideas and guiding the students toward disciplinary ideas.

During the four lessons on analytic geometry, Chris made vid-
eo recordings to observe how Anna orchestrated the discussions.
Analysis of video recordings from the lectures revealed three main
changes in Anna's role. First, the way she reacted to correct or in-
correct solution methods shifted from confirming or setting aside
suggestions, toward making the solution methods the subject of
discussion. Second, the distribution of turns changed such that
more students were involved in the discourse and in reacting to
each other's solution methods. Third, her actions shifted from con-
vergent, teacher-led actions toward divergent, student-led actions.
These results show that within four lessons, an important step has
been taken toward establishing a discourse community.

Thinking like the student
During his research, Chris also explored the decision-making pat-
terns of five experienced Dutch mathematics teachers during their
novice attempts at orchestrating whole-class discourse. His goal
was to understand how teachers try to make sense of the students' ideas and thoughts. During whole-class discussions teachers need
to adapt to the perspective of the students, they need to imagine
what the students' line of thinking could be, so they can help each
student build upon, or change something in their line of thinking.
How teachers understand the thoughts of the students influences
their actions during whole-class discussions. The mathematical ex-
perience of teachers plays an important role in this process. Using
video recordings of the lessons they reflected upon and discussed
the whole-class discussions, and how the teachers and the stu-
dents reacted to certain situations.

Chris grouped the observed teacher sense-making-actions in
categories concerning the role of teacher's mathematical think-
ing in their sense-making: flexibility, preoccupation, incompre-
prehension, exemplification, and projection. Flexibility refers to the ability
of a teacher's mathematical thinking to discuss openly various perspec-
tives of a problem and to consider different solution methods
proposed by students. On the other hand, preoccupation refers to
the situation the content of teachers' mathematical thinking is lim-
lited to one solution method or a particular expected answer, this
inflexibility can impede them in adopting a student's perspective
that doesn't match their own thinking.

Incomprehension describes the situation a teacher does not
mathematically understand a student's solution method or idea,
this can impede sense-making: if the incomprehension does not get
resolved, the teacher is not able to fully investigate the stu-
dent's meaning and reasoning. However, such incomprehension
can also foster adopting the perspective of other students in the
classroom. The process that the teachers' mathematical thinking
can serve as an example of the mathematical thinking they would
like to develop in their students is described as exemplification.
It can support teachers in articulating what kind of thinking they
aim to develop in their students as well as in recognizing to what
extent students already articulate such thinking. Finally, projec-
tion refers to teachers projecting their own ways of mathematical
thinking onto their students, without considering the differences
between their own expert thinking and the thinking of learners,
this can impede them in adopting the students' perspective.

These categories show how both the content and the process
of teacher mathematical thinking can support or impede students'
sense-making. Chris's findings suggest that sense-making of stu-
dents' mathematical thinking requires teachers to (re-)engage in
reflective thinking regarding the mathematical content as well as
the process of their own mathematical thinking.

The more personal aspect
As a final note we would like to give the word to the doctor.

Chris, how did you get interested in mathematics education?
"During my mathematics studies, I regularly provided tutoring and
exam training for high school students and tutorials at the faculty, and
becoming a teacher was sort of a plan B for me. When I ac-
tively started to orient myself on career options during my research
master, I realized I would prefer to stand in front of young people
day all over sitting behind a computer. One month after I had
started teaching, I spoke to Wim Veldman, who had supervised my
master thesis, and he told me there would be a PhD position in
mathematics education. I had never realized that was something
one could do a PhD in! For several years, I combined my teaching
job with PhD research in mathematics education, and gradually
became more and more aware and sure that the learning and
teaching of mathematics would be the focus of my career."

Some other thoughts you would like to share with the readers?
"My work with a group of mathematics teachers was very import-
ant for my PhD project and opened my eyes to all the interesting
mathematics that is involved in high-school mathematics, once
people really start to think and talk about it. I have learned a lot
of mathematics from these teachers and their students. Furthermore,
I would say that this kind of teaching can also help younger stu-
dents and their teachers, as well as higher mathematics education.
For example, if university mathematics students would be sup-
ported (more) in thinking mathematically and talking about their
thinking, I believe more students would succeed in finishing their
studies (and enjoying mathematics!).

Mathematics Education faces many challenges. The most import-
ant is probably the huge shortage of mathematics teachers. In my
position, I make an effort to recruit potential teachers by showing
mathematics students how teaching mathematics involves so much
more than being able to give good explanations, and that as a
teacher, you learn a lot of mathematics, if you encourage students
to think and talk and if you listen to what they say. It would be great
if we, as mathematicians, would all spread the message that teach-
ing mathematics is a very respectable and mathematical profession.
Another important challenge is teaching students of all ages math-
ematics in such a way that they become aware of and develop their
mathematical thinking and see the use and beauty in mathematic-
s. I hope to continue cooperating with researchers, teachers, and
students at multiple levels of mathematics education in making a
modest contribution to facing both challenges."

Concluding
To summarize, in his research Chris investigated how teachers can
organize their mathematics teaching using whole-class discourse.
He collaborated closely with a group of mathematics teachers and
supported them in developing their lessons. Teachers were stim-
ulated to develop and implement discourse-based teaching while
staying close to their practice. They met regularly to discuss and
reflect upon the lesson recordings, and they tried to unravel the
complex process of how teachers try to make sense of the thoughts
and ideas of the students. It is beautiful to see such collabora-
tions flourishing, we hope to see similar projects in the future! This
project shows what is possible when enthusiastic people join their
forces. We wish Chris the best with his future work!