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Research

The arithmetic-geometric mean: A pearl of Gauss

The arithmetic-geometric mean (AGM) algorithm has an amazingly fast convergence. In this article Piet Van Mieghem follows Gauss on his remarkable path of mathematical discoveries, posthumously published in Gauss's Nachlass in 1866.

Carl Friedrich Gauss (1777–1855) was a titan of science [5]. Words fall short to describe his phenomenal mathematical creations: just as paintings and musical pieces by the greatest artists [1], his mathematics fills an impressive gallery of the finest art. The only difference between music and paintings compared to mathematical art is that the latter requires more effort to understand, before its penetrating light embraces human emotions. After all, just as in music and paintings, it requires much technical skills, before creations and art occur. Art is all about emotion. The most beautiful mathematical art shines by its simplicity, which often shields its depth. It may sounds odd that I speak about mathematical art, while most people associate mathematics with a cool and logical system, void of any human emotion. And, yet, there is an ocean of beauty in which the logical pieces are built towards a magnificent castle.

The sequel here is devoted to one type of painting, one style of Gaussian symphony. In [15], I have tried to unravel his unpublished work, posthumously collected in his Nachlass (in *Gauss Werke*, Band 3), about the arithmetic-geometric mean (AGM). This article is a summary of [15] and I will refer to it for details. While the AGM algorithm, explained in formula (2) in the next section, is rather basic and before Gauss discovered by another genius Langrange, it was Gauss, who created an astonishing piece of art. The first part in the Nachlass [8] in Latin is the easiest, because it is sufficiently well explained. That first part also shows Gauss's trajectory towards his first fundamental result (8) via elegant Taylor series expansions. The second part [9] in German is challenging and difficult, because Gauss has merely left sketches or just a list of formulae without any clue nor derivation. Of course, we cannot blame Gauss: he never found the time to publish his work on AGM in his near to perfect style, based on his adagium [2] "pauca, sed matura". After Gauss has seen the work of Abel and Jacobi, who found independently parts of his own discoveries about thirty years later, Gauss seemed to have been content that Abel and Jacobi had relieved him from publishing his work on elliptic functions.

Gauss's path towards the valley of elliptic functions, starting with the AGM algorithm, is extremely hard; a narrow, steep route through high and icy mountains, which today is abandoned, because only the most experienced alpinists may succeed. The valley of elliptic functions, which are doubly periodic functions in the complex plane, is now approached by two avenues. First, elliptic functions are beautifully introduced by Tannery and Molk [12, last chapter], who started from Weierstrass's great work on entire functions. Entire functions do not contain singularities in the finite complex plane, but an essential singularity at infinity, and can be regarded as generalizations of polynomials. Tannery and Molk [13] related Weierstrass's to Jacobi's approach, the latter based on the surprising theta-functions (which Gauss has discovered first, see [15, Section 7]). A second avenue starts from Cauchy's integral theorem and Liouville's elegant theorem, stating that a bounded analytic function without singularities (e.g. poles) in the complex plane is a constant. When applying these two theorems to the fundamental parallelogram of an elliptic functions, many properties are deduced (see e.g. [16]). Yet, I believe that the start of Gauss's path, namely the arithmetic-geometric mean, is worth to explore, essentially due to its amazingly fast convergence.

The arithmetic-geometric mean (AGM)

1. Definition of the arithmetic-geometric mean $M(a,b)$. The arithmetic mean of two numbers a and b is defined by $m_A = \frac{a+b}{2}$ and their geometric mean by $m_G = \sqrt{ab}$. We assume that a and b are non-negative real numbers, to avoid complications with the squareroot in $m_G = \sqrt{ab}$. An immediate bound $m_A \geq m_G$, with equality only if $a = b$, follows from

$$0 \leq \frac{1}{2}\left(\sqrt{a} - \sqrt{b}\right)^2 = \frac{a+b}{2} - \sqrt{ab}. \qquad (1)$$

Gauss [8] studies the sequence $\{(a_n, b_n)\}_{n \geq 0}$, where [3]

$$a_n = \frac{a_{n-1} + b_{n-1}}{2} \text{ and } b_n = \sqrt{a_{n-1} b_{n-1}} \qquad (2)$$

starting from $(a_0, b_0) = (a, b)$, which he already knew at the age of 14 years old [4]. Explicitly, the arithmetic-geometric mean (AGM) algorithm (2), written in two columns, is

$$
\begin{aligned}
a_0 &= a & b_0 &= b\\
a_1 &= \frac{a+b}{2} & b_1 &= \sqrt{ab}\\
a_2 &= \tfrac{1}{2}\left(\tfrac{a+b}{2}+\sqrt{ab}\right) & b_2 &= \sqrt{\tfrac{a+b}{2}\sqrt{ab}}\\
a_3 &= \cdots & b_3 &= \cdots
\end{aligned}
$$

Invoking the inequality $m_A \geq m_G$ to (2) illustrates that $a_n \geq b_n$ for any integer $n \geq 1$. In other words, for $n \geq 1$, the left column with $\{a_n\}_{n\geq 1}$ will contain numbers that are always larger than the right column with $\{b_n\}_{n\geq 1}$, if we exclude the uninteresting case that $a = b$, for which $a_n = b_n = a$ and nothing changes with n. If $n = 0$, we obviously have that $a_0 < b_0$ if $a < b$, but for $n \geq 1$, it holds that $a_n \geq b_n$. In the sequel, therefore, we assume that $a > b$, so that the inequality $a_n > b_n$ holds for any integer $n \geq 0$. Combining $a_n > b_n$ with the arithmetic mean $a_n = \frac{a_{n-1}+b_{n-1}}{2} < a_{n-1}$ then shows, for any integer $n \geq 1$, that $a_n < a_{n-1}$, while the geometric mean $b_n = \sqrt{a_{n-1}b_{n-1}} > b_{n-1}$ shows that $b_n > b_{n-1}$.

2. Convergence of the AGM algorithm in (2). Gauss observes, for $a \geq b \geq 0$, that

$$
\begin{aligned}
\frac{a_n-b_n}{a_{n-1}-b_{n-1}} &= \frac{(a_n-b_n)(a_n+b_n)}{(a_{n-1}-b_{n-1})(a_n+b_n)}\\
&= \frac{a_n^2-b_n^2}{(a_{n-1}-b_{n-1})(a_n+b_n)}\\
&= \frac{\left(\frac{a_{n-1}+b_{n-1}}{2}\right)^2-a_{n-1}b_{n-1}}{(a_{n-1}-b_{n-1})(a_n+b_n)}\\
&= \frac{(a_{n-1}-b_{n-1})^2}{4(a_{n-1}-b_{n-1})(a_n+b_n)}\\
&= \frac{a_{n-1}-b_{n-1}}{2(a_{n-1}+b_{n-1})+4b_n}\\
&\leq \frac{1}{2}\frac{a_{n-1}-b_{n-1}}{a_{n-1}+b_{n-1}} \leq \frac{1}{2}
\end{aligned}
$$

where equality only holds if $b = 0$. Only if $b = 0$, the AGM algorithm (2) reduces to $b_n = 0$ and $a_n = \frac{1}{2}a_{n-1}$ with solution $a_n = \frac{a}{2^n}$ for $n \geq 0$. Hence, for $a > b > 0$, Gauss obtains the inequality $a_n - b_n < \frac{1}{2}(a_{n-1}-b_{n-1})$, which after iteration on $n \geq 0$ shows that

$$a_n - b_n < \frac{1}{2^n}(a-b). \tag{3}$$

With $a_n \pm b_n = \frac{1}{2}(\sqrt{a_{n-1}} \pm \sqrt{b_{n-1}})^2$, we have

$$
\begin{aligned}
\frac{a_n-b_n}{a_n+b_n} &= \left(\frac{\sqrt{a_{n-1}}-\sqrt{b_{n-1}}}{\sqrt{a_{n-1}}+\sqrt{b_{n-1}}}\right)^2\\
&= \left(\frac{a_{n-1}-b_{n-1}}{(\sqrt{a_{n-1}}+\sqrt{b_{n-1}})^2}\right)^2\\
&< \left(\frac{a_{n-1}-b_{n-1}}{a_{n-1}+b_{n-1}}\right)^2,
\end{aligned}
$$

because $(\sqrt{a_{n-1}}+\sqrt{b_{n-1}})^4 = (a_{n-1}+b_{n-1} + 2\sqrt{a_{n-1}b_{n-1}})^2 > (a_{n-1}+b_{n-1})^2$ for $a > b > 0$. Iterated p times,

$$
\begin{aligned}
\frac{a_n-b_n}{a_n+b_n} &< \left(\frac{a_{n-1}-b_{n-1}}{a_{n-1}+b_{n-1}}\right)^2\\
&< \cdots < \left(\frac{a_{n-p}-b_{n-p}}{a_{n-p}+b_{n-p}}\right)^{2^p}
\end{aligned}
$$

leads, after choosing $p = n$,

$$\frac{a_n-b_n}{a_n+b_n} < \left(\frac{a-b}{a+b}\right)^{2^n}. \tag{4}$$

Since $a_{n+1} = 2(a_n+b_n)$ and $a_{n+1} < a_n$, we find that $a_n - b_n < 2a_1\left(\frac{a-b}{a+b}\right)^{2^n}$, which tends to zero considerably faster than $\frac{a-b}{2^n}$ in (3) for $n > n_0$, where n_0 is a threshold value. The AGM algorithm (2) converges quadratically or the convergence of the sequence $\{(a_n-b_n)\}_{n\geq 0}$ is of second order. A convergence of second order means that each iteration in the AGM algorithm (2) for positive a and b approximately doubles the number of correct decimal digits.

In summary, the difference $a_n - b_n$ in the sequence $\{(a_n,b_n)\}_{n\geq 0}$ tends to zero with $n \to \infty$. In other words, the sequences $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ converge to the same limit $M(a,b)$, which Gauss calls the *arithmetic-geometric mean* (AGM),

$$M(a,b) = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n. \tag{5}$$

For $a > b > 0$, the above analysis shows that

$$
\begin{aligned}
a > a_1 > \cdots > a_n > a_{n+1} \geq \cdots \geq M(a,b)\\
\geq \cdots \geq b_{n+1} > b_n > \cdots > b_1 > b,
\end{aligned}
$$

while

$$M(a,a) = a, \qquad M(a,0) = 0.$$

Since any pair (a_n,b_n) in the sequence $\{(a_n,b_n)\}_{n\geq 0}$ converges to the same limit, we also conclude that

$$M(a,b) = M(a_1,b_1) = \cdots = M(a_n,b_n) = \cdots. \tag{6}$$

3. Scaling of $M(a,b)$. If we multiply both a_n and b_n by a positive real number β, then the AGM recursion (2) shows that also a_{n+1} and b_{n+1} are multiplied by β. Hence, $\lim_{n\to\infty} \beta a_n = \beta M(a,b)$ and

$$M(a\beta, b\beta) = \beta M(a,b). \tag{7}$$

Taking $\beta = \frac{1}{a}$ and subsequently $\beta = \frac{1}{b}$ in (7) yields $M(a,b) = aM(1,\frac{b}{a}) = bM(\frac{a}{b},1)$. The scaling (7) means that the study of $M(a,b)$ can be reduced to $M(1,x)$, where $0 \leq x \leq 1$, because we assume as Gauss that $a > b$. However, interchanging a and b in the iterative algorithm (2) does not impact the limit, i.e. $M(a,b) = M(b,a)$. Thus, alternatively $M(a,b)$ can be reduced to $M(x,1)$, where $x \geq 1$. Figure 1 draws $M(x,1)$ together with its upper bound $\frac{1+x}{2}$ and lower bound $\sqrt{x}$ on a lin-lin and log-log scale.

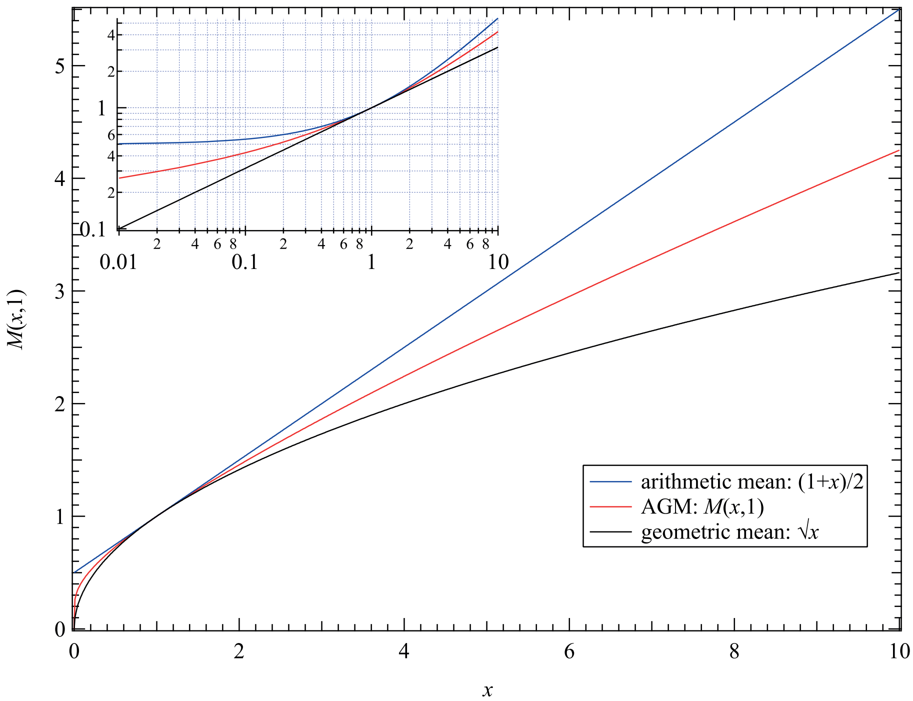

Figure 1 The three means of x and 1: the arithmetic mean $(1+x)/2$, Gauss's AGM $M(x,1)$ and the geometric mean $\sqrt{x}$, both on lin-lin and log-log scale (inset).

Gauss [8, art. 3] computes four examples with different (a,b) up to 20 decimal digits (!) to illustrate how fast the recursion (2) converges. Numerical computations confirm a convergence of second order and that the iteration n in (2) has about twice the number of correct digits than the iteration $n-1$, which is numerically an amazingly fast convergence. This very fast convergence of the sequence in (2) likely attracted Gauss to explore the properties of the arithmetic-geometric mean $M(a,b)$.

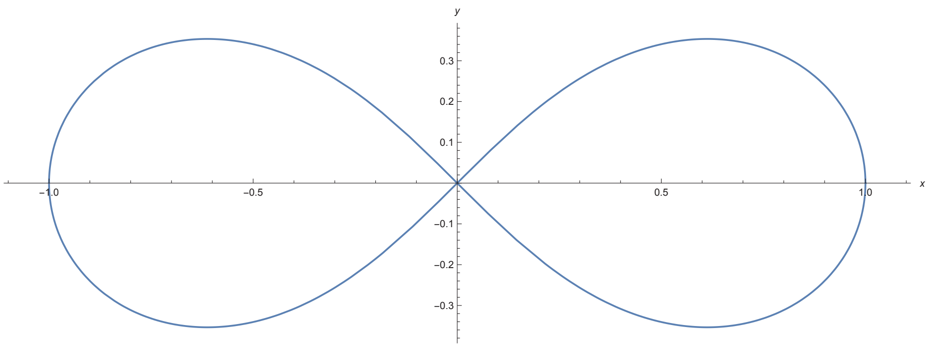

Figure 2 The lemniscate is defined in polar coordinates (r,θ) as $r^2 = \cos(2\theta)$.

Highlights of Gauss's journey

Gauss [8, art. 5] computes several Taylor expansions, like $M(1+x,1) = \sum_{k=0}^{\infty} h_k x^k$, but each time concludes that the Taylor coefficients do not satisfy an interesting law. Eventually, he deduces the Taylor series

$$\begin{aligned}\frac{1}{M(1+x,1-x)} &= \sum_{m=0}^{\infty}\left(\frac{(2m)!}{(2^m m!)^2}\right)^2 x^{2m} \\ &= \sum_{m=0}^{\infty}\binom{\frac{1}{2}}{m}^2 x^{2m} \\ &= 1+\frac{1}{4}x^2+\frac{9}{64}x^4+\frac{25}{256}x^6+\frac{1225}{16384}x^8+\dots\end{aligned}$$

which can also be written in terms of Gauss's hypergeometric function as

$$\frac{1}{M(1+x,1-x)} = F\left(\tfrac{1}{2},\tfrac{1}{2},1;x^2\right).$$

Gauss wrote a great manuscript [7] on the hypergeometric function, which basically generalized Newton's working horse, the binomial expansion $(1+x)^\alpha = \sum_{k=0}^{\infty}\binom{\alpha}{k}x^k$ and which derives the theory of the Gamma function most elegantly (that led Weierstrass later to his artful theory of entire functions). Hence, Gauss rapidly saw that the Taylor expansion can be written as an integral,

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}} = \frac{\pi}{2}\frac{1}{M(a,b)} \quad (8)$$

called a 'tour de force' by McKean and Moll [11].

Gauss's fundamental integral (8) is rewritten in terms of Legendre's complete elliptic integral

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

of the first kind as

$$\frac{1}{a}K\left(\frac{1}{a}\sqrt{a^2-b^2}\right) = \frac{\pi}{2}\frac{1}{M(a,b)}.$$

Gauss knew the theory of lemniscate, invented by Jacob Bernoulli in 1694, and realized that the total length L_l of the lemniscate equals

$$L_l = \frac{2\pi}{M(\sqrt{2},1)}.$$

Consequently, a basic result in Gauss's investigations [4, p. 280–283] relates the total length of the lemniscate to the arithmetic-geometric mean

$$\begin{aligned}M(\sqrt{2},1) &= \frac{\pi}{2\int_0^1 \frac{dy}{\sqrt{1-y^4}}} \\ &= \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4}\right)} \simeq 1.19814.\end{aligned}$$

Gauss denotes $\varpi = 2\int_0^1 \frac{dy}{\sqrt{1-y^4}} \simeq 2.39628$ to emphasize the importance of $L_l = \frac{2\pi}{M(\sqrt{2},1)}$ which is then

$$M(\sqrt{2},1) = \frac{\pi}{\varpi}. \quad (9)$$

Since the integral $\int_0^x \frac{dy}{\sqrt{1-y^4}}$ resembles $\int_0^x \frac{dy}{\sqrt{1-y^2}} = \arcsin(x)$, Gauss studied the inverse function of $\int_0^x \frac{dy}{\sqrt{1-y^4}}$, because the properties of $\sin(x)$ are much more elegant than those of its inverse function $\arcsin(x)$. Gauss [4] defined the lemniscatic functions as

$$\operatorname{sinlem}\left(\int_0^x \frac{dy}{\sqrt{1-y^4}}\right) = x$$

and deduced many beautiful series expansion and theta-function like expansions, that were never published by Gauss, only in his Nachlass. Gauss's sinus and cosinus lemniscatus are a special case of Jacobi's elliptic amplitudinis functions [10] and doubly-periodic with periods 2ϖ and $2i\varpi$. Cox [4, Section 3] demonstrates that Gauss had a complete theory of elliptic functions!

Computations of π

The number π has fascinated humans for over 4000 years since the Babylonians and old-Egyptians and is still captivating current mathematicians. The Borwein brothers and Bailey [3] overview the history of the computing π. We will only discuss here Archimedes' algorithm, Leibniz' and Euler's series and the application of Gauss's AGM algorithm.

4. *Archimedes' computation of* π. Surprisingly similar to Gauss's AGM recursion (2), the Borwein brothers [2] (see also [6, pp. 31–35] mention Archimedes' recursion

$$A_n = \frac{A_{n-1}+B_n}{2} \text{ and } B_n = \sqrt{A_{n-1}B_{n-1}} \quad (10)$$

where $\frac{1}{A_n}$ is the area of the circumscribed regular 2^n-gon and $\frac{1}{B_n}$ denotes the area of an inscribed regular 2^n-gon around a circle with radius 1. The recursion of Archimedes (ca. 287–212 BC) in (10) seems due to Gauss's teacher Pfaff [3, p. 205] and is derived in [15]. Comparing with the circle, we find the inequalities $\frac{1}{B_n} < \pi < \frac{1}{A_n}$ and the recursion (10), starting at $n=2$ with $B_2 = \frac{1}{2}$ and $A_2 = \frac{1}{4}$, converges to π. Table 1 computes Archimedes' recursion (10) for π up to $n=15$.

5. *Leibniz's series*. Perhaps the simplest or most classic series to compute π are derived from inverse trigonometric functions. We confine ourselves to series for $\arctan z$, whose Taylor series around $z=0$ is

$$\arctan z = \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}z^{2k+1} \quad \text{for } |z|<1, \quad (11)$$

which converges for $z=1$, because Leibniz' series

$$\begin{aligned}\frac{\pi}{4} &= \lim_{K\to\infty}\sum_{k=0}^{K}\frac{(-1)^k}{2k+1} \\ &= 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\frac{1}{15}+\dots \quad (12)\end{aligned}$$

is an alternating sum with decreasing terms. Most likely, Leibniz' series (12) is one of the simplest, but also slowest con-

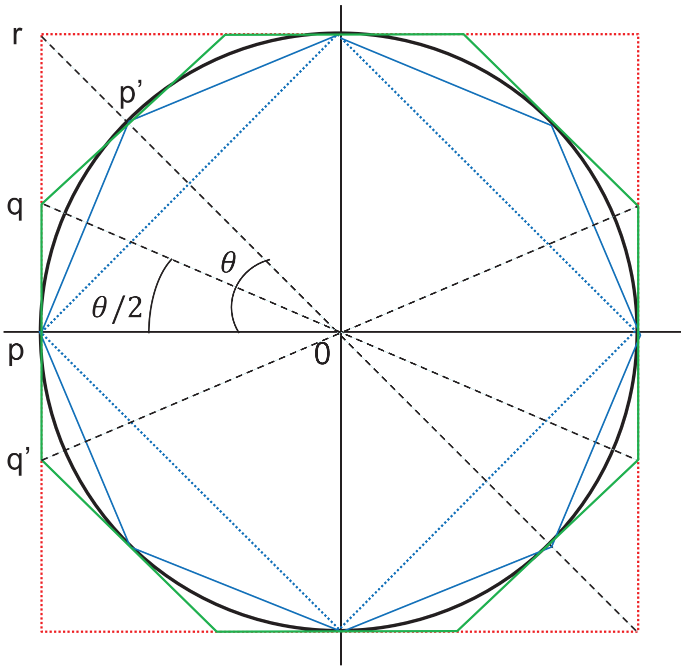

Figure 3 Inscribed and circumscribed 2^n-gon of a circle with radius 1.

n	$\frac{1}{A_n}$	$\frac{1}{B_n}$	$\frac{1}{A_n}-\frac{1}{B_n}$
1	4	2	2
2	3.3137084989	2.82842712474	0.485281
3	3.1825978780	3.06146745892	0.12113
4	3.1517249074	3.12144515225	0.0302798
5	3.1441183852	3.13654849054	0.00756989
6	3.1422236299	3.14033115695	0.00189247
7	3.1417503691	3.14127725093	0.000473118
8	3.1416320807	3.14151380114	0.00011828
9	3.1416025102	3.14157294036	0.0000295699
10	3.1415951177	3.14158772527	$7.39247\ 10^{-6}$
11	3.1415932696	3.14159142151	$1.84812\ 10^{-6}$
12	3.1415928075	3.14159234557	$4.6203\ 10^{-7}$
13	3.1415926920	3.14159257658	$1.15507\ 10^{-7}$
14	3.1415926632	3.14159263433	$2.88768\ 10^{-8}$
15	3.1415926559	3.14159264877	$7.21921\ 10^{-9}$

Table 1 Archimedes' recursion (10) for π up to $n=15$.

vergent series for π. If $K=10^m$ terms are computed in (12), then about m decimal digits are correct. Leibniz did not obtain (12) as derived above, but from his general method of 'transmutation' , which is nicely explained by Edwards [6, p. 245–252]. The companion series of (12), due to Newton and equally slowly converging, is

$$\frac{\pi}{2\sqrt{2}}=\lim_{K\to\infty}\sum_{k=0}^{K}(-1)^k\left(\frac{1}{4k+1}+\frac{1}{4k+3}\right)$$
$$=1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{13}-\frac{1}{15}+\cdots. \tag{13}$$

Newton's π-series (13) is computed in [15].

Many other variations on $\arctan z$ exists [2, p. 352]. John Machin (1680–1752) found that

$$\pi=16\arctan\left(\frac{1}{5}\right)-4\arctan\left(\frac{1}{239}\right),$$

while Leonhard Euler (1707-1783) started from

$$\pi=20\arctan\left(\frac{1}{7}\right)+8\arctan\left(\frac{3}{79}\right).$$

Euler introduced his famous Euler transform and derived [14] the series in 1755,

$$\arctan z$$
$$=\frac{z}{1+z^2}\lim_{K\to\infty}\sum_{k=0}^{K}\frac{(k!)^2 2^{2k}}{(2k+1)!}\left(\frac{z^2}{1+z^2}\right)^k \tag{14}$$
$$=\frac{z}{1+z^2}\left(1+\frac{2}{3}y+\frac{2\cdot 4}{3\cdot 5}y^2+\frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}y^3+\cdots\right)_{y=\frac{z^2}{1+z^2}}.$$

Borwein and Borwein [2] mention that Euler computed π to 20 decimal places in an hour, because powers of $\left.\frac{z^2}{1+z^2}\right|_{z=1/7}=\frac{1}{50}$ and $\left.\frac{z^2}{1+z^2}\right|_{z=3/79}=\frac{9}{6250}$ are small.

6. AGM computation of π. Almkvist and Berndt [1, Theorem 5] mention

$$\pi=\frac{4M^2(1,\frac{1}{\sqrt{2}})}{1-\sum_{n=1}^{\infty}2^{n+1}c_n^2} \tag{15}$$

where $c_n=\sqrt{a_n^2-b_n^2}$. The entire derivation of (15) is in [15]. Borwein and Borwein [1, Section 5] present another algorithm for π with second order convergence.

Only 10 iterations in (15), with $a=1$ and $b=\frac{1}{\sqrt{2}}$, lead to an astonishingly small error at $n=9$ of less than $10^{-2^9}\approx 10^{-512}$, as illustrated in Table 2, where 30 decimal digits are given.

We again observe that, at each iteration in n, the number of decimal digits approximately doubles!!

n	a_n	b_n	$2c_{n+1}=a_n-b_n$	formula (15)
0	1	0.707106781186547524400844362105	0.29289	4
1	0.853553390593273762200422181052	0.840896415253714543031125476233	0.012656	3.187672642712108627201929970530
2	0.847224902923494152615773828643	0.847201266746891460403631453693	0.000023636	3.141680293297653293918070424560
3	0.847213084835192806509702641168	0.847213084752765366704298051780	$8.24274\ 10^{-11}$	3.141592653895446496002914758820
4	0.847213084793979086607000346474	0.847213084793979086605997900490	$1.00244\ 10^{-21}$	3.141592653589793238466360602710
5	0.847213084793979086606499123482	0.847213084793979086606499123482	$1.48265\ 10^{-43}$	3.141592653589793238462643383280
6			$3.24336\ 10^{-87}$	
7			$1.55206\ 10^{-174}$	
8			$3.55415\ 10^{-349}$	
9			$1.86375\ 10^{-698}$	

Table 2 The 30 decimal digits AGM computation for π in formula (15) up to iteration $n=9$.

n	$\frac{1}{2^{n-1}}\log\left(\frac{4b_{n+1}}{c_n}\right)$	$\frac{1}{2^{n-1}}\log\left(\frac{4a_n}{c_n}\right)$
0	3.11916231251975389237754454656	3.46573590279972654708616060729
1	3.14157172949917097506914830630	3.14904153515897666929968289288
2	3.14159265355330856833419360289	3.14159962823802109942254203509
3	3.14159265358979323846242152809	3.14159265360195479517183083648
4	3.14159265358979323846264338328	3.14159265358979323846271733501

Table 3 The first four iterations of formula (16).

We add another AGM computation of π due to Gauss [9, p. 377], and which did not appear in recent literature,

$$\frac{1}{2^n}\log\left(\frac{4b_{n+1}}{c_n}\right) \le \frac{\pi}{2}\frac{M(a,b)}{M(a,c)} \le \frac{1}{2^n}\log\left(\frac{4a_n}{c_n}\right). \quad (16)$$

If $a = b\sqrt{2} = c\sqrt{2}$, then $\frac{M(a,b)}{M(a,c)} = 1$ and, confining to $n = 0$ in (16), we have that

$$\log\left(\frac{4a_0}{c_0}\right) = \log(4\sqrt{2}) = \frac{5}{2}\log(2),$$

while

$$\log\left(\frac{4b_1}{c_0}\right) = \log\left(\frac{4\sqrt{a_0 b_0}}{c_0}\right) = \log\left(2^{2+\frac{1}{4}}\right) = \frac{9}{4}\log 2$$

from which Gauss concludes that

$$\frac{9}{4}\log 2 < \frac{\pi}{2} < \frac{10}{4}\log 2.$$

If n increases in (16), we find lower and upper bounds for π where each step in n results in the famous approximate doubling of decimal digits. Choosing $a = 1$ and $b = \frac{1}{\sqrt{2}}$ as above, the first four iterations of (16) are as shown in Table 3, while for $n = 5$ all 30 digits are correct. The lower bound for $n = 4$ has already 30 decimals correct, while the upper bound only has 22 correct decimals. Hence, we observe that Gauss's upper and lower bounds (16) converge a little faster in n towards π than (15), in spite of the computation of the logarithm, which is numerically more demanding.

Summary

The extremely fast convergence of the AGM algorithm (2) is still attractive in times of the abundant presence of computers. Except for inverse functions computed by Newton–Raphson's method (that is also of second order), it is still unknown [2, p. 352] whether there are other functions than complete elliptic integrals that possess a quadratically converging algorithm such as Gauss's AGM algorithm (12).

Notes

1 Ludwig Van Beethoven (1770–1827), with Flemish roots from Mechelen, was a German contemporary of Carl Friedrich Gauss.

2 "Pauca, sed matura" is Latin and means "Few, but ripe". Gauss wrote clearly and briefly in a Ciceroan Latin style. He avoided unnecessary prose, in contrast to the then ruling French scientists of the Academy Francaise in Paris. Each sentence in his work plays a role; it is hard to further condense or skip parts without missing the idea. When contempories complained to him that he shielded the way in which he has found his discoveries, Gauss briefly replied: "Have you ever seen a beautiful building, to which the scaffold is still attached?" 'Pauca' does not imply that Gauss wrote only few articles. In fact, he was very productive, but he could have published more if his high writing standard was reduced. His work on AGM is an example; he did not publish this pearl.

3 Gauss writes accents instead of subscripts in n; thus $a' = a_1$, $a'' = a_2$, $a''' = a_3$, et cetera.

4 *Gauss Werke*, Band 3, p. 404 on 'Elegantiores integralis $\int_0^x \frac{dy}{\sqrt{1-y^4}}$ proprietatis', in which he defines the sinuslemniscatus and derives many of its functional properties, much more than the sinus possesses.

References

1 G. Almkvist and B.C. Berndt, Gauss, Landen, Ramanuyan, the arithmic-geometric mean, Ellipses, π and the Ladies Diary, *American Mathematical Monthly* 95 (1988), 585–608.

2 J.M. Borwein and P.B. Borwein, The arithmic-geometric mean and fast computation of elementary functions, *SIAM Review* 26(3) (1984), 351-366.

3 J.M. Borwein, P.B. Borwein and D.H. Bailey, Ramanujan, modular equations and approximations to pi or how to compute one billion digits of pi, *The American Mathematical Monthly* 96(3) (1989), 201–219.

4 D.A. Cox, The arithmetic-geometric mean of Gauss, *L'Enseignement Mathématique* 30 (1984), 275–330.

5 G.W. Dunnington, *Carl Friedrich Gauss: Titan of Science*, The Mathematical Association of America, 2004. Reprinted version, first edition by Hafner Publishing, 1955.

6 C.H. Edwards Jr., *The Historical Development of the Calculus*, Springer, 1979.

7 C.F. Gauss, Disquisitiones generales circa seriem infinitam $1 + \frac{\alpha\beta}{1\cdot\gamma}x + \frac{\alpha(1+\alpha)\beta(1+\beta)}{1\cdot 2\gamma(\gamma+1)}xx + \frac{\alpha(1+\alpha)(2+\alpha)\beta(1+\beta)(2+\beta)}{1\cdot 2\cdot 3\gamma(\gamma+1)(2+\gamma)}x^3$ etc., pars prior, *Commentationes societatis regiae scientiarum Gottingensis recentiores*, Vol. II, in *Gauss Werke*, Band III, 1813, pp. 125–162.

8 C.F. Gauss, De origine proprietatibusque generalibus numerorum mediorum arithmeticogeometricorum, *Nachlass* in *Gauss Werke*, Band 3, 1866, pp. 362–374.

9 C.F. Gauss, Fortsetzung der Untersuchungen über das Arithmetisch Geometrische Mittel, *Nachlass* in *Gauss Werke*, Band 3, 1866, pp. 377–403.

10 C.G.J. Jacobi, *Fundamenta Nova Theoriae Functionum Ellipticarum*, Borntraeger, 1829.

11 H. McKean and V. Moll, *Elliptic Curves: Function Theory, Geometry, Arithmetic*, Cambridge University Press, 1997.

12 J. Tannery and J. Molk, *Éléments de la Théorie des Functions Elliptiques*, Thome I, Gauthier-Villars et Fils, 1893.

13 J. Tannery and J. Molk, *Éléments de la Théorie des Functions Elliptiques*, Thome II, Gauthier-Villars et Fils, 1896.

14 P. Van Mieghem, Characteristic coefficients of a complex function (1993–2020), unpublished.

15 P. Van Mieghem, *The Arithmetic-Geometric Mean: A pearl of Gauss*, Delft University of Technology, 2023, Report 20230605.

16 E.T. Whittaker and G.N. Watson, *A Course of Modern Analysis*, Cambridge University Press, 1996.