Problemen

| Problem Section

This Problem Section is open to everyone; everybody is encouraged to send in solutions and propose problems. Group contributions are welcome. We will select the most elegant solutions for publication. For this, solutions should be received before **15 April 2023**. The solutions of the problems in this issue will appear in one of the subsequent issues.

Problem A

Does there exist a partitioning X of $\mathbb{R}$ into infinite sets such that for every *choice map* $c: X \to \mathbb{R}$, i.e. a map c such that $c(S) \in S$ for all $S \in X$, the image of c is dense in $\mathbb{R}$?

Problem B

Show that for all $k \in \mathbb{Z}$ there exists an $x \in \mathbb{Q}$ for which there are at least two subsets $S \subseteq \mathbb{Z}_{\geq 1}$ such that $\sum_{s \in S} s^k = x$.

Problem C (proposed by Daan van Gent)

For a group G and $g \in G$ write $c(g) = \{hgh^{-1} \mid h \in G\}$ and $G^\circ = \{g \in G \mid \#c(g) < \infty\}$.

a. Show that G° is a normal subgroup of G and that $G^{\circ\circ} = G^\circ$.

b. Now define $G_\circ = G/G^\circ$. Show that there exists a group G for which the sequence $G, G_\circ, G_{\circ\circ}, \ldots$ does not stabilize, i.e. for none of the groups H in the sequence we have $H^\circ = 1$.

Edition 2022-3 We received solutions from Rik Biel, Brian Gilding and Pieter de Groen.

Problem 2022-3/A

1. Let $n \in \mathbb{Z}_{\geq 1}$ and let $f: \mathbb{R}^n \to \mathbb{R}^n$ be continuous such that for all $x \in \mathbb{R}^n \setminus \{0\}$ we have $|f(x)| < |x|$. Write f^m for the mth iteration of f. Prove that
$$\lim_{m \to \infty} f^m(x) = 0.$$
2. Denote by ℓ^2 the Hilbert space of square-summable sequences of real numbers. Prove that there exists a continuous map $f: \ell^2 \to \ell^2$ such that for all $x \in \ell^2$ we have $|f(x)| < |x|$ and for some $a \in \ell^2$ we have that $\{f^m(a)\}_{m=1}^\infty$ does not converge.

Solution This problem is solved by Brian Gilding, Pieter de Groen and partially solved by Rik Biel. This proof is due to Brian Gilding.

1. It suffices to show that for all $x \in \mathbb{R}^n$ and $\varepsilon > 0$ there exists an $m \geq 1$ such that $|f^m(x)| < \varepsilon$. If $|x| \leq \varepsilon$, this holds for $m = 1$. On the other hand, if $|x| > \varepsilon$, then, by the compactness of $S := \{z \in \mathbb{R}^n : \varepsilon \leq |z| \leq |x|\}$ and by continuity of f, the map $S \to [0,1)$ given by $z \mapsto |f(z)|/|z|$ attains a maximum $k \in [0,1)$ on S. Hence, $|f^m(x)| \leq k^m |x|$ for all $m \geq 1$ for which $|f^{m-1}(x)| \geq \varepsilon$. This gives $|f^m(x)| < \varepsilon$ for sufficiently large m.
2. Consider
$$f(x) = (0, g(1,x_1), g(2,x_2), g(3,x_3), \ldots) \quad \text{for } x = (x_1, x_2, x_3, \ldots),$$
where
$$g(n,t) = \frac{n(n+2)}{(n+1)^2} t.$$

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Oplossingen | Solutions

For all $x \in \ell^2$ we have $f(x) \in \ell^2$, since $g(n,\cdot)$ is a contraction on $\mathbb{R}$ for every $n \geq 1$. Moreover, for all $x \in \ell^2$ we have $|f(x)| \leq |x|$, with strict inequality if $x \neq 0$. Inasmuch f is linear, it follows that $f : \ell^2 \to \ell^2$ is continuous. Defining $e_1 = (1,0,0,0,\ldots)$, $e_2 = (0,1,0,0,0,\ldots)$, $e_3 = (0,0,1,0,0,0,\ldots)$ and so on,

$$f^m(e_i) = \left(\prod_{j=i}^{m+i-1} g(j,1)\right) e_{m+i} = \frac{i(m+i+1)}{(i+1)(m+i)} e_{m+i} \quad \text{for every } m \geq 1 \text{ and } i \geq 1.$$

Thus, for all $a \in \ell^2 \backslash \{0\}$, $\{f^m(a)\}_{m=1}^{\infty}$ has no accumulation points in ℓ^2.

Problem 2022-3/B

Prove that for every integer n there exists a finite group G such that n equals the number of normal subgroups minus the number of non-normal subgroups.

Solution For a group G write $s(G) = (a,b)$ where a is the number of normal subgroups and b the number of non-normal subgroups of G. For all $k \in \mathbb{Z}_{\geq 0}$ and p prime we have $s(\mathbb{Z}/p^k\mathbb{Z}) = (k+1,0)$, so for all $n > 0$ we are done. For $n = 0$ we notice that $s(S_3) = (3,3)$.

Claim. Let G_1 and G_2 be finite groups of coprime order with $s(G_i) = (a_i, b_i)$. Then $s(G_1 \times G_2) = (a_1 a_2, a_1 b_2 + a_2 b_1 + b_1 b_2)$.

Proof. Let $n = \#G_1$ and $m = \#G_2$. Then by Bézout there exist $x,y \in \mathbb{Z}$ such that $xn + ym = 1$. Let $H \subseteq G_1 \times G_2$ be a subgroup. For $(g,h) \in H$ we have

$$H \ni (g,h)^{ym} = (g^{ym}, h^{ym}) = (g^{1-xn}, 1) = (g,1),$$

and similarly $(1,h) \in H$. Hence $H = H_1 \times H_2$ for subgroups $H_i \subseteq G_i$. Note that H is normal if and only if H_i is normal in G_i for both i. □

For $n < 0$ it suffices to find a finite group G with $s(G) = (a,b)$ and $a - b = -1$, since $s(G \times (\mathbb{Z}/p^k\mathbb{Z})) = (a(k+1), b(k+1))$ and $a(k+1) - b(k+1) = -(k+1)$ for $p \nmid \#G$ a prime and $k \in \mathbb{Z}_{\geq 0}$.

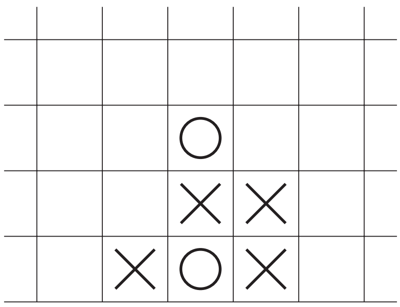
Consider the semidihedral group

$$\mathrm{SD}_{16} = \langle a, x \mid a^8 = x^2 = 1,\, xax^{-1} = a^3 \rangle = C_8 \rtimes C_2$$

of order 16. A subgroup $H \subseteq \mathrm{SD}_{16}$ either contains a^4, or is of the form $\{1\}$ or $\langle a^{2k}x \rangle$ for $k \in \mathbb{Z}/4\mathbb{Z}$. In the former cases we may interpret H as a subgroup of the quotient $\mathrm{SD}_{16}/\langle a^4 \rangle \cong \mathrm{D}_8$ with $s(\mathrm{D}_8) = (6,4)$, while in the latter case only $\{1\}$ is normal. Hence $s(\mathrm{SD}_{16}) = (7,8)$.

Problem 2022-3/C

Olivia and Xavier play the game *Connect Three* on an infinite half grid on a sheet of paper. The rules are as follows: Olivia and Xavier take alternating turns, starting with Olivia. In her turn, Olivia draws an ○ in a square with no empty squares below. In Xavier's turn, he twice draws an × in a square with no empty squares below. Olivia wins if she gets three ○'s in a row, either horizontally, vertically, or, diagonally. Can Xavier prevent Olivia from winning?



Solution Yes. Represent the game board by $\mathbb{Z}\times\mathbb{Z}_{\geq 0}$. Then Xavier plays according to the following rules:

1. Whenever Olivia plays (a,b) with $a\equiv 0\,(3)$ or $b>0$, Xavier plays $(a,b+1)$ and $(a,b+2)$.
2. Whenever Olivia plays $(a,0)$ with $a\not\equiv 0\,(3)$, Xavier plays $(v,0)$, where $v>a$ is minimal such that $v\not\equiv 0\,(3)$ and $(v,0)$ is empty. Moreover, Xavier plays $(u,0)$, where $u<a$ is maximal such that $u\not\equiv 0\,(3)$ and $(u,0)$ is empty.

Let $a\in\mathbb{Z}$. One inductively shows that after Xavier's turn

1. if $a\equiv 0\,(3)$, then column a has height $\equiv 0\,(3)$, with (a,b) containing an $\times$ precisely when $b\not\equiv 0\,(3)$;
2. if $a\not\equiv 0\,(3)$, the column a is either empty or has height $\not\equiv 0\,(3)$, with (a,b) containing an $\times$ when $b\equiv 2\,(3)$.

By 1 and 2 no $\bigcirc$ will every be placed in a row b with $b\equiv 2\,(3)$. Hence Olivia cannot obtain a vertical or diagonal three-in-a-row. By 1 no horizontal three-in-a-row can be obtained in a row b with $b\not\equiv 0\,(3)$, while by 2 no horizontal three-in-a-row can be obtained in a row b with $b\equiv 0$ and $b>0$. Finally, note that it is impossible that $(a,0)$ and $(a+1,0)$ both contain an $\bigcirc$ for $a\equiv 1\,(3)$. The first time Olivia plays in either of these squares, the other either already contains an $\times$, or it is empty, after which Xavier will play in it by rule 2. Hence Olivia cannot win.

Note: In the September 2022 issue Thijmen Krebs should also have been mentioned as one of the contributors of solutions for Problems 2022-1/A, 2022-1/B and 2022-1/C.