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**History** Lviv, Ukraine

# The Scottish Book

**The ongoing tragedy in Ukraine brings to mind a piece of mathematical cultural heritage from almost a century ago when Lviv, a city on the western borders of the country, was a center of excellence that hosted some of the most well known analysts and topologists of that time. In this article Raf Bocklandt describes the story of this flourishing mathematical community and the famous relic that it brought forth: the Scottish Book.**

## Lviv

Before the war in Ukraine started, Lviv, previously also known as Lwów in Polish or Lemberg in German, was a well kept secret that featured prominently on the bucket list of connoisseur city trippers. This hidden beauty spot was one of the few cities in Eastern Europe whose historic center has largely remained intact, despite its occupation by the Germans in the Second World War. Even now because it lies at the western border of the country it has largely been spared from the devastation that eastern cities like Charkiv and Mariupol suffered.

Nowadays Lviv is part of Ukraine, but it has seen many different rulers during the almost eight hundred years of its existence. It started out as the capital of the long since vanished kingdom of Ruthenia, a medieval eastern Slavic state that covered parts of Poland, Belarus, Moldova, Romania and Ukraine. In the fourteenth century it became part of Poland and this rule lasted for four hundred years until the Polish monarchy was split up between Prussia, Russia and Austria at the end of the eighteenth century.

Illustration: Wikimedia Commons

Lviv on a map of modern day Ukraine

Lwów ended up in Austrian hands and was renamed Lemberg. During the nineteenth century the city grew into the fourth largest city in the Austrian empire. At first the Austrians tried to Germanize the city, but after the empire was restyled to the dual monarchy Austria-Hungary, the city gained more independence and became an important cultural center for Polish, Ukrainian and Jewish people. With the break-up of the empire after the First World War, both the Poles and Ukrainians tried to win over the city. A civil war ensued, and after two years of conflict Lwów became part of the reestablished Polish state.

During the twenties and thirties the city became the cradle for the cultural rebirth of the Polish nation. Counting five institutions of higher learning and many academies and societies, it had a very lively intellectual community and the streets and bars were buzzing with excitement. During that time the Lwów School of Mathematics was established under the impulse of Hugo Steinhaus and Stefan Banach.

## The Scottish Café

Steinhaus [2, 5] was born in 1887 in a Jewish family and had studied mathematics and philosophy at the University of Lemberg (as it was called in these days). After a year he moved to Göttingen, did a PhD with Hilbert and in 1917 he returned to his alma mater where he later became a full professor. Between finishing his PhD and getting a permanent academic position he spent some time in Krakow. There, walking in a park, he overheard someone talking about Lesbesgue integration, a hot topic in analysis but not really a standard theme for casual causeries. Intrigued by the strange conversation, Steinhaus introduced himself to the two people sitting on the bench. One of them was Stefan Banach, a brilliant mind who had never bothered to take formal exams but maintained an active interest in current mathematical research.

After their serendipitous meeting in Krakow, Steinhaus and Banach become good friends, despite having very different characters. Steinhaus was a classical cultured professor, fluent in Latin and Greek, who gave minutiously prepared but sometimes difficult to follow lectures. He had broad interests: he wrote about a lot of different topics in geometry and analysis, and collaborated with economists, physicians and biologists. Banach [3, 6] on the other hand was a single-minded genius whose research focused almost solely on functional analysis. To the dismay of his later colleagues he didn't care about the proper dress professorial code, was often short of money, showed up late during term, and made up his — nevertheless inspiring — classes on the fly. On a mathematical level their different styles complemented each other and this proved to be a perfect match. They not only worked together on various mathematical problems, but they also founded a new journal on functional analysis and a series of mathematical monographs.

Steinhaus quickly realized the ingenuity of his new friend and helped him to obtain an academic post in Lwów despite his lack of formal qualifications. The two friends, now settled at the same university, became the

Photo: Wikimedia Commons

Banach with his friend Nikodym discussing Lesbesgue integration on a bench in Krakow.

Photo: WikiPedia

A present day picture of the Scottish Café

Photo: MacTutor History of Mathematics Archive

Stanisław Mazur hands over a goose to Per Enflo as a reward for solving problem 153 of the Scottish Book.

core of a larger group of mathematicians, which included other famous names such as Stanisław Ulam, Stanisław Mazur, Mark Kac, Kazimierz Kuratowski, Julius Schauder and Stefan Kaczmarz. Together they formed a local chapter of the Polish Mathematical Society, which held weekly meetings on Saturday, after which they would frequent some of the bars near the department.

Banach liked to do mathematics in bars. Fueled by coffee, cognac, beer and cigarettes no one could keep up with his fast thinking and drinking. On the Academic Square not far from the university stood a peculiar building with an arched portal flanked by two plump turrets. Inside on the ground floor resided the Scottish Café, a coffee house decorated in the Viennese style with marble tables. It wasn't the bar with the most refined food and drinks, but because of the owner's generous bar tab policy it became Banach's favourite hangout place and with him he brought along a flock of fellow mathematicians. He would always be the first one to arrive and the last to leave, only to return the next day with handwritten proofs of the problems discussed the previous day.

It did not take long before they started meeting on a daily basis, using the marble table tops as writing boards to accompany their mathematical discussions. This was not an ideal solution as it did not please the owner and often their scribbles were washed away the following day. In 1935 Banach's wife came up with an idea, she bought them a big notebook that was kept safely behind the counter of the bar and which could be consulted by any visitor who felt a mathematical urge.

The *Scottish Book*, as it was called, was not only reserved for the local mathematicians, but visitors from Krakow, Warsaw or even from places abroad such as Paris or Moscow were encouraged to contribute when they visited the department. Over the years the book filled with mathematical problems, both easy and difficult. Some were serious research questions, others just fun recreational puzzles. Some even came with prizes, like an ounce of caviar, a bottle of whiskey, or a live goose.

**One problem for lunch**

Many of the problems in the Scottish Book were concerned with the areas of expertise of the group: point-set topology, measure theory and functional analysis [7]. These were fairly new fields at that time: the now classical definition of a topological space had barely been settled upon, the Lesbesgue measure dated from the beginning of the century, while the notion of a Banach space — a vector space with a complete norm — a crucial ingredient for functional analysis, was being developed in Lwów at that very moment by our main protagonist. These new abstract concepts enabled them to replicate classical analytical arguments to show that certain maps had zeros or fixed points, which had many applications in geometry, analysis and the study of differential equations.

As an example, let us focus on one of the problems from the book itself: nr 123. Although originally formulated quite technically, the result became later known in its colloquial culinary version, the ham sandwich theorem [1]. In the problem, Steinhaus

Photo: MacTutor History of Mathematics Archive

Hugo Steinhaus

Photo: MacTutor History of Mathematics Archive

Stefan Banach

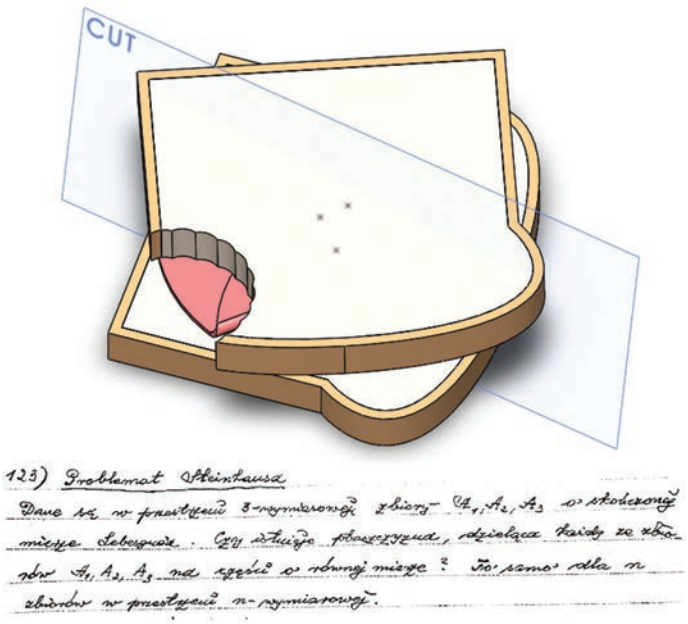


Illustration: https://www.lastsandwich.com/articles/culture-sandwich/61
Description: http://kielich.amu.edu.pl/Stefan_Banach/pdf/ks-szkocka/ks-szkocka1pol.pdf

The ham sandwich problem with its original description from the Scottish Book in Polish

asked whether it is possible to cut each configuration of three subsets of $\mathbb{R}^3$ with finite Lesbesgue measure (or, if you want, two slices of bread and a slice of ham) by a plane such that each subset is divided precisely in two halves of equal size.

It was Banach who first came up with a solution [8] using another famous theorem involving mathematicians who wrote problems in the Scottish Book: the Borsuk–Ulam theorem. Stanisław Ulam was a student in Lwów who, apart from being an accomplished topologist also later became notorious for being the coinventer of the hydrogen bomb. He had a friend in Warsaw, Karol Borsuk, who sometimes came over to meet him and join the others in the Scottish Café. Influenced by these meetings, Borsuk published a proof of a conjecture by Ulam, which now bears their names [4]. It states that every continuous map $f:\mathbb{S}^n \to \mathbb{R}^n$ from the $n$-sphere to $n$-dimensional space for which $f(-x)=-f(x)$ has $0$ in its image.

The ham sandwich theorem — and its generalization involving an $n$-dimensional hypersandwich with $n-2$ condiments — is a nice consequence of Borsuk's result. Using the intermediate value theorem, one can find for each unit vector a plane normal to it that cuts the slice of ham exactly in half. This plane does not necessarily cut the two slices of bread in equal halves. Measure the differences between the upper and lower parts of the other two slices and gather these two differences in a vector. This gives a continuous map from the unit sphere to the plane. It satisfies the additional requirement $f(-x)=-f(x)$ because the opposite unit vector corresponds to the same plane, but reverses the role of the upper and lower part. The unit vector which maps to zero comes from a plane for which the differences are both zero, in other words, where both slices of bread are cut in equal halves.

## Aftermath

In September of 1939 Poland was again split up and Lwów was annexed by the Soviet Union. The university was Ukrainized and renamed after Ivan Franko, a Ukrainian writer. Some of the Polish mathematicians left, but Banach, who kept good relations with both Russians and Ukrainians, stayed and even became dean of the science faculty. The remaining mathematicians kept on meeting in the Scottish Café and even received visits from well known Soviet mathematicians such as Sobolev, Bogolyubov and Alexandroff, who left their marks in some of the final problems of the book.

In 1941 Hitler declared war on Stalin and the city was conquered by the Germans. Of its 120 000 Jewish inhabitants only about 800 would survive the war, but also its Polish population was hit hard. Many intellectuals and university professors were shot on the spot or sent to camps. Only half of the members of the mathematics group pulled through the war unscathed. The head of the department was executed, the Jew Julius Schauder was never seen again, and Banach survived under harsh conditions, but died of lung cancer in 1945. After the war, the boundaries of Poland were shifted to the west and Lviv stayed in the Ukrainian part of the Soviet Union. The Polish population was forced out and Banach's wife smuggled the Scottish Book to Wrocław. Later Stanisław Ulam, who had fled to the United States before the war, translated it to English and published it [7].

The original book stayed in the hands of Banach's family, but a copy still can be found on display in the Scottish Café — now restyled as Restaurant 'Szkocka' — and offers the mathematically oriented visitors some extra brain food to supplement their dinner. In recent years the mathematical department of the University of Lviv has even installed a new version of the Scottish Book, which true to the modern age now not only has a paper version but also a web page [9], a Facebook account, a YouTube channel, and a MathOverflow user-id to keep the word wide mathematical community up to date with its new problems.

## References

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