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Research

(Mind the) gaps between primes

The question of whether there are infinitely many pairs of 'twin primes' (primes that differ by 2) has puzzled mathematicians for hundreds, if not thousands, of years. Until recently, it was not even known whether there are infinitely many pairs of primes that differ by a finite number. In 2013, Yitang Zhang stunned the mathematics community by proving that there are infinitely many pairs of primes that differ by at most 70 000 000. While 70 000 000 is still quite far from 2, Zhang's work has inspired a flurry of activity on this problem, leading to many other interesting results in number theory. This expository article is based on a talk that Lola Thompson gave with the same title at the Winter Symposium of the Koninklijk Wiskundig Genootschap (Royal Dutch Mathematical Society) on 9 January 2021. The other speaker was Pieter Moree, who also has an article in this issue.

Patterns in the prime numbers?

Prime numbers have long been a source of fascination. Just as some people would look to the skies and wonder about the stars, others would look at lists of whole numbers and wonder about the primes: that is, those numbers greater than 1 that are only divisible by 1 and themselves. Like DNA in biological organisms, primes can be viewed as the 'building blocks' of the integers. Every positive integer greater than 1 can be expressed uniquely as a product of primes. In other words, the prime factorization of an integer completely specifies that integer.

Some would say that the primes behave erratically, seeming to appear at random places, while others would say that they display a surprising amount of symmetry.

For example, in Figure 1 the positive whole numbers are arranged in a spiral, starting with 1 at the center. There is a black dot at each prime that appears on the line. Remarkably, the black dots appear to arrange themselves along diagonal lines. Perhaps there is a certain rhythm of the primes after all!

Looking at the primes naturally leads one to look for patterns. However, one has to be a bit careful when looking for patterns in the primes. In 1650, Fermat famously conjectured that all numbers of the form $2^{2^n}+1$ (where $n=0,1,2,3,4,\dots$) are prime. He based his conjecture on the fact that the pattern holds for $n=0,\dots,4$, and that was as far as he could compute by hand. However, in 1732, Euler showed that Fermat's conjecture fails when $n=5$. As of 2010, it is known that $2^{2^n}+1$ is composite for $5\le n\le 32$. Some mathematicians have even conjectured that $2^{2^n}+1$ is composite for all $n\ge 5$! If they are correct, then Fermat could not have been more wrong! In short, just because the smaller primes seem to display certain patterns, one cannot extrapolate that these patterns hold in general. One needs to prove, rigorously, that these patterns hold continue past the small data points that humans are able to compute. The fact that the primes appear to land along the diagonals in the spiral above is also a phenomenon that dissi-

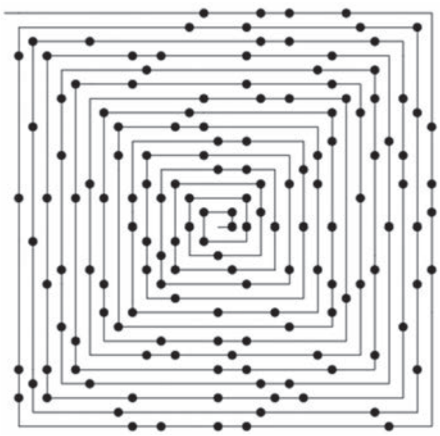

Figure 1 Prime spiral, by Eric W. Weisstein [15].

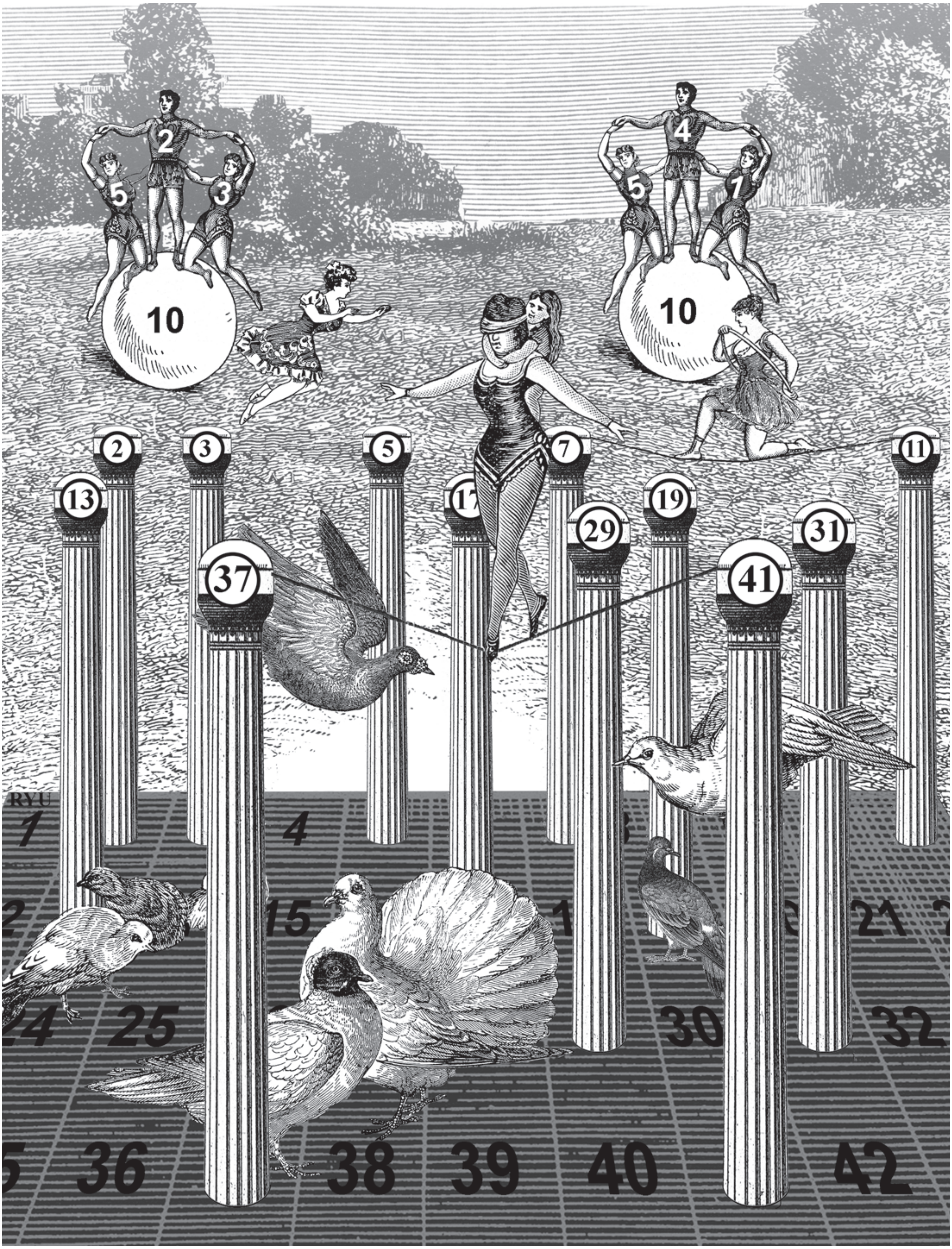

Illustration: Ryu Tajiri

1	**2**	**3**	4	**5**	6
7	8	9	10	**11**	12
13	14	15	16	**17**	18
19	20	21	22	**23**	24
25	26	27	28	**29**	30
31	32	33	34	35	36
37	38	39	40	**41**	42
43	44	45	46	**47**	48
49	50	51	52	**53**	54

Table 1

pates as you zoom out, though there is a good reason for it.

As another cautionary tale, consider the positive whole numbers in Table 1. The casual observer might notice that all of the primes (except 2) seem to fall in columns '1' and '5'. Unlike the previous example, this pattern actually holds in general. One could prove that all primes greater than 2 fall into columns '1' and '5' if an infinite table were constructed in the same manner. Unfortunately, this is not a deep observation. Notice that, every entry in column '1' can be written in the form $6n+1$, where $n=0,1,2,\ldots$. Similarly, column 2 corresponds to the numbers of the form $6n+2$, and the other whole numbers can be written in the form $6n+3$, $6n+4$, $6n+5$, or $6n$. Notice that $6n+2=2(3n+1)$. This means that $6n+2$ can always be factored into a product of two numbers that are between 1 and $6n+2$, so it cannot be prime. Similar logic shows that $6n+3$ and $6n+4$ also cannot be prime. Moreover, $6n$ is clearly not prime since it is a product of 6 and another number. As a result, the only columns that could possibly contain prime numbers are columns '1' and '5' above. The fact that we have convinced ourselves that the pattern in the table is true in only a few sentences shows that this pattern is pretty trivial.

In this article, we will focus on some patterns in the primes that are both provably true and nontrivial.

Prime number theory

How many primes are there? It turns out that there are infinitely many of them. The first few prime numbers are relatively close together: $2,3,5,7,\ldots$. However, as you zoom out further, the primes seem to spread out. Indeed, there are four prime numbers between 1 and 10, in other words 40% of the first ten positive numbers are prime. However, only 25% of the first one hundred numbers turn out to be prime, and the percentage appears to get worse as you look at larger and larger intervals. One can see this phenomenon illustrated in the following Table 2.

One might look at this table and wonder, in the long run, what percentage of the positive whole numbers are prime. Perhaps surprisingly, as you make the table infinitely large, the percentage will approach 0%! In other words, there are infinitely many primes, but they are extremely rare within the set of positive whole numbers. The primes are so rare that, if you put all positive whole numbers into a paper bag and draw one number out at random, you would have a 0% chance of selecting a prime!

10	4	40%
100	25	25%
1 000	168	16.8%
10 000	1 229	12.29%
100 000	9 592	9.592%
1 000 000	78 498	7.85%
10 000 000	664 579	6.65%
100 000 000	5 761 455	5.70%
1 000 000 000	50 847 534	5.09%
10 000 000 000	455 052 511	4.55%
100 000 000 000	4 118 054 813	4.12%
1 000 000 000 000	37 607 912 018	3.77%

Table 2 Percentage of numbers that are prime.

Twin primes

One pattern that does seem promising in the exploration of prime numbers is the incidence of *twin primes*: primes that differ by 2. The following are pairs of twin primes: $(3,5)$, $(5,7)$, $(11,13)$,... Some very large pairs of twin primes have been found! For example, the numbers

$$(3756801695685\cdot 2^{666669}-1$$
$$3756801695685\cdot 2^{666669}+1)$$

are both prime and they only differ by 2, making them a twin prime pair. However, it is not yet known whether there are infinitely many such pairs. Determining whether there are infinitely many pairs of twin primes is one of many famous unsolved problems in number theory.

More generally, one could ask if there are infinitely many pairs of primes that differ by any positive integer h that you like. This question can be answered very easily when h is odd. Since there is only one even prime number, namely 2, then we only get odd differences between primes when the smaller prime is 2. As a result, gaps of size h for odd h can appear at most once. The situation where h is even appears to be a lot more interesting. In the special case where $h=2$, these are the pairs of twin primes that we considered above. However, we could also look at primes that differ by 4 or 6 or 8 or...

We refer to differences between consecutive primes as 'gaps' between primes. It has been conjectured that all even gaps between primes occur infinitely often:

Conjecture 1 (de Polignac, 1849). *For even integers h, there are infinitely many pairs of primes $p,p+h$.*

This is the first known appearance of the so-called 'twin primes conjecture' that appears in the literature. While the statement seems simple enough, it is astonishingly difficult to prove. In fact, up until recently, one could not even show that any particular integer appears infinitely often as a gap between primes! We still do not know of a single example of a prime gap that occurs infinitely often. However, as we will see below, it has been shown (as of 2013) that at least one finite number appears infinitely often as a gap between primes.

The GPY approach

In 2003, Dan Goldston and Cem Yıldırım announced a proof that there are infinitely many pairs of primes that differ by at most 12. In other words, they claimed that they could prove that one of the numbers 2, 4, 6, 8, 10 or 12 appears infinitely often as a gap between primes. This announcement was met with a great deal of excitement. After all, it would be the first time that anyone could prove that a finite number appears infinitely often as a gap between primes! Unfortunately, their work was quickly discredited by Granville and Soundararajan, who found a fatal flaw in their argument. While Goldston and Yıldırım were unable to salvage the original result, they were able to join forces with another author, János Pintz, to obtain a *conditional* result. A conditional result is one that depends on the validity of a different unsolved conjecture. In their case, they proved:

Theorem 1 (Goldston, Pintz and Yıldırım, 2005). *If a certain unsolved conjecture is true, then there are infinitely many pairs of primes that differ by at most* 16.

(The unsolved conjecture is called the Elliot–Halberstam Conjecture.) On the surface, this only looks a bit worse than what they initially announced. After all, it would just mean that one of the numbers 2, 4, 6, 8, 10, 12, 14 or 16 appears infinitely often as a gap between primes. However, the famous unsolved conjecture that they assumed is also likely to be very difficult to prove. For that reason, the result is somewhat less satisfying than if it were an *unconditional* proof.

For several years, the general consensus in the number theory community was that Goldston, Pintz and Yıldırım had pushed the existing mathematical tools as far as possible. In other words, it would take a whole new set of tools in order to prove that there are infinitely many pairs of primes with a finite number appearing as a gap between them. Then, out of nowhere, a relatively unknown mathematician named Yitang Zhang announced that he had done what had long eluded number theorists: he had shown that there are infinitely many pairs of primes that differ by at most $70\,000\,000$. By the pigeonhole principle this means that there must be some finite (even) number between 2 and $70\,000\,000$ that appears infinitely often as a gap between primes! (The pigeonhole principle says that if you have $n+1$ pigeons that you want to stuff into n holes, there must be at least one hole containing two pigeons. Moreover, if you have infinitely many pigeons that you want to stuff into finitely many holes, then there must be a hole with an infinite number of pigeons. In our example, the "pigeons' are gaps between pairs of primes and the 'holes' are the even numbers between 2 and $70\,000\,000$.

Zhang's approach

How did Zhang manage to overcome the difficulties that Goldston, Pintz and Yıldırım encountered?

When his result was announced, he seemed like an unlikely superhero. After earning his PhD, he was unable to secure an academic position. Instead, he spent five years doing a series of odd jobs (Sandwich Artist at a Subway restaurant, motel employee in Kentucky, delivery worker in a New York City restaurant) before taking an adjunct position at the University of New Hampshire. Prior to his seminal paper on gaps between primes, he had only written two other papers, including his PhD thesis which was never published. He was already in his late 50's when he made his groundbreaking discovery. Perhaps working outside of the traditional academic system is what gave him the freedom to follow his convictions. He believed that, rather than needing to develop a whole new set of tools, he could simply relax some conditions in Goldston, Pintz and Yıldırım's work in order to overcome the barrier that prevented them from obtaining an unconditional result.

The barrier in Goldston, Pintz and Yıldırım's paper had to do with how 'uniformly distributed' the primes are. In order to understand what is meant by this, we will need to take a step back and discuss *modular arithmetic*. Simply put, modular arithmetic is something that we do every day when we tell time on an analogue clock. For example, when the digital clock says that it is $13{:}00$, we mentally convert it to the time on the old analogue clock and say that it is 1 PM. Similarly, $14{:}00$ corresponds to 2 PM, $15{:}00$ corresponds to 3 PM, et cetera.

Instead of creating clock faces with twelve numbers, we can create clock faces with any quantity of numbers that we like. We could make a clock with only three numbers, or we could make a clock with thirteen numbers. For various reasons, number theorists like to work with clocks that have a prime number of numbers.

In the example shown in Figure 2, we see that the numbers keep wrapping around the clock. Since 7 behaves like midnight in this clock, it means that 8 corresponds to 1, and 9 corresponds to $2,\ldots$ but we can also imagine traveling around the clock in a counterclockwise fashion and have a notion of negative numbers on our clock. We say that the numbers on a clock with

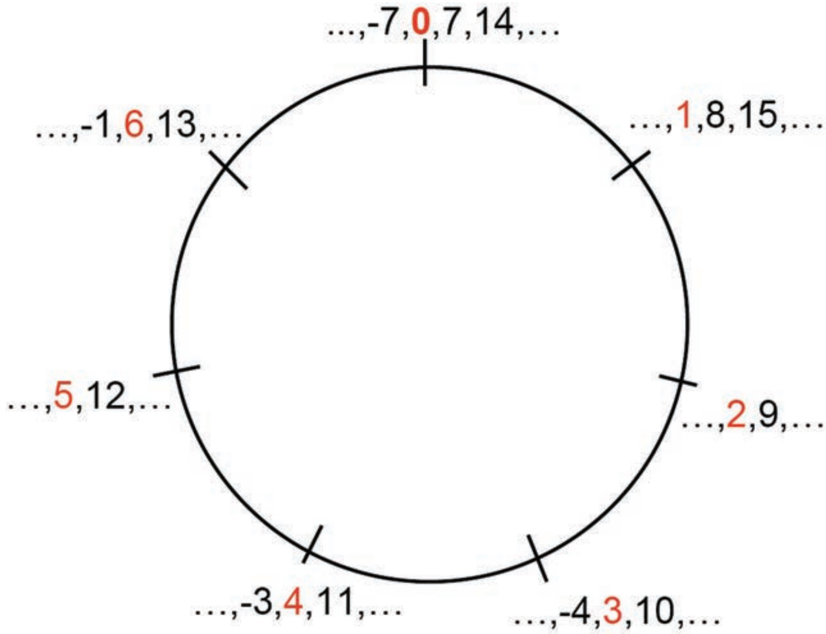

Figure 2 Modulo 7 clock by Richard Taylor [14].

7 values are integers (mod 7). When two numbers are equivalent to one another on the clock with 7 numbers, we say that they are congruent (mod 7). For example, we just saw that $8 \equiv 1 \pmod 7$. The picture also shows us that $-1 \equiv 6 \pmod 7$. When two numbers are equivalent in this clock arithmetic, we say that they fall into the same *residue class*.

In order to prove an unconditional version of their result, Goldston, Pintz and Yıldırım (GPY) would have needed to show that the primes are reasonably 'well-distributed' among the residue classes (mod q) for all positive integers q up to a certain point. As an example, the primes are 'well-distributed' $(\bmod\ 7)$ if there is an equal proportion of primes in each possible residue class (it turns out that primes other than 7 cannot be congruent to $0 \pmod 7$ since this would imply that they are divisible by 7 and we know that primes have no divisors besides 1 and themselves). This leaves six possible residue classes that the primes can fall into, namely they can be congruent to any of the numbers between 1 and $6 \pmod 7$. So, being reasonably well-distributed would mean that about $1/6$ of the primes are congruent to $1 \pmod 7$, roughly $1/6$ are congruent to $2 \pmod 7$, et cetera. It has been proven that this is true when the size of the clock is relatively small. However, when looking over a range of numbers $[1,N]$, the GPY proof relies on the assumption that the primes are reasonably well distributed modulo q when q is just a tiny bit larger than $N^{1/2}$. It turns out that the primes are well-distributed for q smaller than $N^{1/2}$; nevertheless, one cannot prove unconditionally that the same phenomenon occurs for $q > N^{1/2}$. To get past this issue, Zhang put some extra restrictions on the values of q that he looked at, but was able to show that for integers q with these restrictions, he could prove that the primes were reasonably well-distributed, even when q is a bit larger than $N^{1/2}$. In spite of these extra restrictions on q, it was enough to push the GPY machinery through and obtain the following unconditional result:

Theorem 2 (Zhang, May 2013). *There are infinitely many pairs of primes that are at most* $70\,000\,000$ *apart.*

At long last, the world had confirmation that there is a finite number that appears infinitely often as a gap between primes!

Maynard and Tao's approach

When hearing about twin primes, students often ask about the next logical case: can there be infinitely many triples of primes? Here, the answer depends on what is meant by a 'triple of primes'. If one looks at triples of the form $(p, p+2, p+4)$ and asks whether they can be prime infinitely often, the answer is, unfortunately, 'no'. The reason that this cannot happen is because one of the numbers p, $p+2$, and $p+4$ must always be divisible by 3. As an exercise, you can use clock arithmetic to check this for yourself! (Hint: Consider three cases: $p \equiv 1 \pmod 3$, $p \equiv 2 \pmod 3$, and $p \equiv 0 \pmod 3$, and then see what this tells you about $p+2$ and $p+4$ in each case.)

In order to exclude the cases of tuples whose entries *cannot* all be prime, we define the notion of an admissible k-tuple:

Definition 1. We say that a k-tuple $(h_1, \ldots, h_k)$ of nonnegative integers is *admissible* if it doesn't cover all of the possible residue classes $(\bmod\ p)$ for any prime p.

For example, $(0,2,6,8,12)$ is an admissible 5-tuple. We can see this by checking to make sure that, for every prime p, there are always some residue classes $(\bmod\ p)$ that are not covered by this tuple. Notice that none of the terms in the 5-tuple are congruent to $1 \pmod 2$, so the residue 1 is not covered. Similarly, all of the numbers in the 5-tuple are congruent to 2 and $0 \pmod 3$, so the residue 1 remains uncovered. One can check this for all small prime clock sizes and then, once you have more possible residue classes than terms in the tuple, it is clear that there will always be some residue classes that are not covered (in this example, it is sufficient to check through $p = 5$, since after that there will be more than 5 residue classes $(\bmod\ p)$).

Now that we have excluded the impossible prime tuples by establishing an admissibility criterion, we can state a nice generalization of gaps between pairs of primes:

Conjecture 2 (Hardy-Littlewood prime k-tuples). *Let* $\mathcal{H} = (h_1, \ldots, h_k)$ *be admissible. Then there are infinitely many integers* n *such that all of* $n + h_1$ *are prime.*

Unfortunately, this is even more difficult to prove, and it remains a conjecture. However, progress in this direction was made a mere six months after Zhang's paper on gaps between primes appeared on the arXiv. James Maynard [7] and Terence Tao independently showed that one could obtain k-tuples that include a certain number of primes infinitely often. Namely, they proved:

Theorem 3 (Maynard-Tao, November 2013). *Let* $m \geq 2$*. For any admissible* k*-tuple* $(h_1, \ldots, h_k)$ *with* k *sufficiently large (relative to the size of* m*), there are infinitely many* n *such that at least* m *of* $n + h_1, \ldots, n + h_k$ *are prime.*

Notice that one cannot tell which of the m numbers in $n + h_1$ wind up being prime. However, this result is already stronger than what Zhang showed, since the $k = 1$ case implies that there are bounded gaps between pairs of primes. The astounding part is that Maynard and Tao used an approach that was completely different from that used by Zhang. Rather than relaxing the conditions on q and showing that the primes are reasonably well-distributed in a slightly broader range, they developed a powerful new method that has already revealed itself to be useful in a broad array of applications.

The method used by Maynard and Tao requires using a mathematical tool called a sieve. Just like a collander is used to separate pasta from water, or a plastic sieve is used on the beach to separate sea shells from sand, a mathematical sieve is used to separate numbers with a property that we desire from numbers without that property. Maynard and Tao pre-sieved the set of positive integers, creating a sample space that consisted only of those that are likely to be prime and closer together than average. Then they were able to show that the weighted average of the number of primes among $n + h_1, \ldots, n + h_k$ over the sample space that they constructed is sufficiently large that there must be a certain number of primes in the tuple.

Polymath improvements

Recall that Zhang's original result showed that there are infinitely many pairs of primes that differ by at most $70\,000\,000$. Almost immediately, various mathematicians set out to whittle that number down. Tao had the brilliant idea of crowdsourcing this work, spreading the tasks over dozens of mathematicians in order to speed up the process. This became known as the polymath project, more specifically Polymath 8, since it was the eighth crowdsourced project to come from this initiative. Using Zhang's approach, Polymath 8 was able to bring the gap size of at most $70\,000\,000$ all the way down to at most 4680. Using the approach of Maynard and Tao, Polymath 8 [11] was able to obtain further improvements. The current state-of-the-art is as follows:

Theorem 4 (D.H.J. Polymath, 2014). *There are infinitely many pairs of primes that are at most* 246 *apart.*

In other words, by the pigeonhole principle, we know that at least one even number between 2 and 246 appears infinitely often as a gap between primes! Moreover, if one is willing to assume an unsolved conjecture (The Elliot–Halberstam Conjecture), Polymath 8 has obtained the conditional result that at least one of the numbers 2, 4, or 6 must appear infinitely often as a gap between primes!

Applications of the Maynard–Tao machinery

In this section, we discuss just a few of the many applications of Maynard and Tao's work on small gaps between primes. The main focus will be on the author's own work, since one of the aims of the Winter Symposium of the Royal Dutch Mathematical Society was to highlight the research of mathematicians working in the Netherlands. In reality, there are many interesting applications of the Maynard–Tao approach, far too many to list here. For a more comprehensive guide, see, for example, the excellent article by Andrew Granville [5].

Prime gaps between polynomials

Just like the primes were the building blocks of the integers, we could also look at building blocks of polynomials: these are called *irreducible polynomials* and they are polynomials that cannot be factored any further. For example, we can factor the polynomial $x^3-1=(x-1)(x^2+x+1)$. As hard as we try, we cannot factor $x-1$ or x^2+x+1 into smaller polynomials. As a result, $x-1$ and x^2+x+1 are irreducible polynomials. In fact, $x-1$ and x^2+x+1 are examples of *cyclotomic polynomials*, which you can learn more about by reading Pieter Moree's article [8] in this issue.)

As one might imagine, there are many parallels between prime numbers and prime polynomials. In fact, there is a whole area of number theory research dedicated to understanding which results about prime numbers still hold for prime polynomials. Thus, it is natural to consider whether we can translate the results of Maynard and Tao about gaps between prime numbers and obtain analogous results for polynomials.

Just like we could reduce integers $(\mathrm{mod}\ p)$, where p is a prime number, we can also do this with polynomials.

Example: The polynomial $4x^2+5x+1$ is irreducible. However, if we reduce it $(\mathrm{mod}\ 3)$ then it factors as follows:

$$\begin{aligned} 4x^2+5x+1 &\equiv x^2+2x+1 \,(\mathrm{mod}\ 3) \\ &\equiv (x+1)^2 \,(\mathrm{mod}\ 3). \end{aligned}$$

Let $\mathbb{Z}_p[x]$ denote the set of polynomials $(\mathrm{mod}\ p)$. The first to prove a 'bounded gaps' result for prime polynomials was Chris Hall, who showed in his PhD thesis [6] that any of the numbers $1,\ldots,p-1$ can occur as gaps between prime polynomials in $\mathbb{Z}_p[x]$ infinitely often, provided that p is greater than 3. The case where $p=3$ turned out to be trickier. However, two years later, Paul Pollack showed in his PhD thesis [9] that the result also holds when $p=3$. Combining their results yields the following theorem that holds for all $p\geq 3$:

Theorem 5 (Hall, 2006; Pollack, 2008). *If* $p\geq 3$, *then any* $a\in\mathbb{Z}_p$ *(excluding* $a=0$*) occurs infinitely often as a gap between irreducible polynomials.*

Notice that this is a much stronger result than anything that we can say about gaps between prime numbers. For example, this result implies that there are infinitely many pairs of prime polynomials that differ by 2. If we could prove that for prime integers, we would solve the famous twin primes conjecture! Indeed, it is often easier to prove the polynomial analogues of famously difficult problems about prime numbers.

After the paper of Maynard [7] appeared on the arXiv, Hall and Pollack teamed up with a few others who were interested in studying the methods of Maynard and Tao. This is where the author of the present article enters the picture. Along with Abel Castillo, Chris Hall, Robert Lemke Oliver, and Paul Pollack, we showed [2]:

Theorem 6 (Castillo, Hall, Lemke Oliver, Pollack and Thompson, 2014). *Let* $m\geq 2$. *For any admissible* k*-tuple* $(h_1,\ldots,h_k)$ *of polynomials in* $\mathbb{Z}_p[x]$ *with* k *'sufficiently large', there are infinitely many* $f\in\mathbb{Z}_p[x]$ *such that at least* m *of* $f+h_1,\ldots,f+h_k$ *are irreducible.*

This improves on the original work of Hall and Pollack in several ways. First, our 'gaps' need not be prime integers: in fact, we also can prove that our result holds when f is any monomial. Second, the proofs of Hall and Pollack are constructive, and the examples of pairs of prime polynomials that differ by a specific integer jump up in degrees very quickly. In other words, their method produces a relatively sparse set of examples of prime polynomials with bounded gaps between them. In contrast, our prime polynomial pairs occur in many degrees. In fact, we can even show that, for sufficiently large degrees, any positive proportion of elements of $\mathbb{Z}_p[x]$ of bounded degree can occur as a gap between prime polynomials!

This work has recently been extended by Sawin and Shusterman in [12], who prove a number of prime polynomial analogues of problems that remain open for prime integers.

Digit sums

One perhaps surprising application of the proof method of Maynard and Tao has to do with digit sums. Notice that, if we take the sum of the digits of 523, we obtain $5+2+3=10$. Similarly, if we sum the digits of 541, then we get $5+4+1=10$. It turns out that 523 and 541 are consecutive primes. It is natural to wonder whether there are other pairs of consecutive primes with the same digit sum in base 10. As always, there are a number of ways to generalize this question. One could look at triples of consecutive primes that have the same digit sum, or quadruples, or quintuples,... Moreover one could consider digit sums in other bases, rather than just looking at base-10 digit sums (as in the example above). From now on, let $s_g(n)$ denote the sum of the base-g digits of n.

In 1961, Sierpiński posed the question: "Are there arbitrarily long runs of consecutive primes p on which $s_g(p)$ is constant?" He also considered the related questions of whether one could find arbitrarily long runs of consecutive primes on which the digit sum is always increasing (or always decreasing). He showed [13] that he could find increasing pairs of digit sums, i.e., $s_{10}(p_n) < s_{10}(p_{n+1})$, infinitely often. The following year, Erdős showed that one can also find decreasing pairs of digit sums infinitely often. In 1968, Sierpiński went a step further, showing that if an unsolved conjecture (Dickson's prime k-tuples conjecture) is assumed, then one can obtain decreasing triples of consecutive primes infinitely often, i.e., $s_{10}(p_n) > s_{10}(p_{n+1}) > s_{10}(p_{n+2})$. Some time later, in an unpublished claim, Schinzel announced that, assuming a different unsolved conjecture (Schinzel's Hypothesis H), one can show that there are arbitrarily long runs of consecutive primes on which the digit sums are increasing, and similarly there are arbitrarily long runs of consecutive primes on which the digit sums are decreasing.

In late 2013, my co-author and I became aware that the methods developed by Maynard and Tao were being used to prove a number of results on runs of consecutive primes. The first such paper was written by Banks, Freiberg and Turnage-Butterbaugh [1] within the same month that Maynard's paper appeared on the arXiv. We realized that it would be possible to use these new methods to generalize the results of Sierpinski, Erdős and Schinzel. In particular, we showed [10]:

Theorem 7 (Pollack and Thompson, 2014). *For any base g, there are arbitrarily long runs of consecutive primes p on which $s_g(p)$ is constant/increasing/decreasing.*

Note that we generalize the prior work on digit sums on consecutive primes in a number of ways. First, our results hold for digits sums in any base (i.e., we are not limited to base 10). Moreover, our results do not depend on the validity of any unsolved conjectures — they are unconditional! The proof method goes beyond the scope of this expository article, but it is worth briefly mentioning that it combines the proof method of Maynard–Tao with a deep result of Drmota, Mauduit and Rivat [3].

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