Raf Bocklandt
Korteweg-de Vries Institute for Mathematics
University of Amsterdam
r.r.j.bocklandt@uva.nl

History

A testimonial of troubled times

Sometimes a theorem is more than just a mathematical statement, it can bear witness of a period in history, not only through its content but also through the mathematicians that are associated to it. The Krull–Remak–Schmidt theorem is a prime example of this. From a mathematical point of view it is a showcase for the rise of abstract algebra in the first half of the twentieth century, while the lives of the three mathematicians behind it got tangled into the political upheaval because of their markedly different backgrounds and the choices they made. In this article Raf Bocklandt gives a short description of this remarkable history.

A modern theorem with an ancient flavour

Splitting mathematical objects into its basic building blocks has been around since Euclid proved the uniqueness of prime factorization, but decomposition theorems only gained prominence in the beginning of the twentieth century with the arrival of abstract algebra. Among these, the Krull–Remak–Schmidt theorem generalizes the prime factorization theorem to the world of abstract groups. In this setting the glue used to stick groups together is the direct product. This construction takes two groups G,H and forms the group $G\times H$ of all pairs of elements with component wise multiplication. The most basic example, known to all students taking a first course in group theory, is Klein's famous four-group, which is the direct product of $\mathbb{Z}_2$ with itself. See Table 1.

The direct product of two groups $G\times H$ contains two special subgroups, $G\times\{e_G\}$ and $\{e_H\}\times H$, which are disjoint apart from the unit element $e=(e_G,e_H)$, commute and generate the whole group. Vice versa, if we can find two proper subgroups in a group meeting these three requirements, then this group is isomorphic to the direct product of these two subgroups. Groups that don't have such subgroups are called *indecomposable* and they form the basic building blocks, analogous to the primes in the prime factorization theorem. For every prime number p there is indeed an indecomposible group of that order: the cyclic group $\mathbb{Z}_p$, but there are a lot more indecomposable groups: the first ones we encounter are $\mathbb{Z}_4$ together with all its cousins of the form $\mathbb{Z}_{p^n}$, the symmetric and dihedral groups, and the quaternion group Q_8.

$+$	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$(0,0)$	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$(0,1)$	$(0,1)$	$(0,0)$	$(1,1)$	$(1,0)$
$(1,0)$	$(1,0)$	$(1,1)$	$(0,0)$	$(0,1)$
$(1,1)$	$(1,1)$	$(1,0)$	$(0,1)$	$(0,0)$

Table 1

So indecomposability is a rather weak condition, but it is enough to show that any finite group can be written as a direct product of indecomposable groups. Moreover just like for prime factorizations this product is unique but here it becomes interesting: what do we precisely mean by unique? For the prime factorization of numbers the uniqueness is clear. If we have two factorizations into primes we can pair the primes occurring in both factorizations such that they are the same. In the case of groups the meaning of the same has to be changed to isomorphic, but it turns out that we can do a bit better.

To illustrate this, let us return to Klein's four-group. This is the product of $\mathbb{Z}_2\times\mathbb{Z}_2$, but if we put this group on the dissection table we see that it contains in fact three subgroups isomorphic to $\mathbb{Z}_2$:

$$G_1 = \{(0,0),(1,0)\},$$
$$G_2 = \{(0,0),(0,1)\},$$
$$G_3 = \{(0,0),(1,1)\}.$$

This means that we can write the four-group in six different ways as a direct product $G = G_i \times G_j$. Because all the G_i are isomorphic to $\mathbb{Z}_2$ these decompositions all look the same. But there is more: the four-group has six automorphisms (corresponding to the six invertible 2×2 matrices with coefficients in $\mathbb{Z}_2$) which work transitively on the pairs $(G_i, G_j)_{i\neq j}$, so the decompositions are the same in a very precise way: they can be turned into each other using automorphisms of the group. (In fact it suffices to use central automorphisms. These are the automorphisms that commute with the inner automorphisms.) This leads to the following formulation of the Krull–Remak–Schmidt theorem for finite groups.

Theorem. *Every finite group can be written as a direct product of indecomposable subgroups and the group of central automorphisms acts transitively on the decompositions.*

A recalcitrant rebel

The history of this theorem can be traced back to Joseph Wedderburn, a Scottish mathematician who wrote the first textbook on abstract group theory. In a short paper of 1909 [4] Wedderburn discusses the notion of direct products of finite groups and states the theorem of the unique decomposition but only in the weak sense: the factors of two decompositions are pairwise isomorphic.

The stronger version dates from two years later and was proved by Robert Remak in his PhD thesis under the supervision of Georg Frobenius, another giant of the theory of finite groups [10]. Robert Remak was born in 1888 in Berlin into a family of Ashkenazy Jews [9]. His grandfather, a renowned embryologist with the same name, was the first Jew in Germany to be appointed as a professor without having to leave his Jewish faith. Robert himself also had academic ambitions, he first studied medicine and mathematics, but after his PhD he had a lot of difficulties to find a position at a university.

Remak was the kind of student that many professors fear in their classes, pretending to be asleep during the lecture, but eager to point out mistakes or imprecise statements as soon as they appeared on the blackboard [1]. Also on the political level he liked to poke holes into the arguments of his opponents and therefore he was often branded a communist, while this was not really correct. Remak was merely interested in the interplay between mathematics and economics and hoped that the math would indicate which politico-economic system was to be preferred. Later he turned these ideas into an essay on the question whether economic studies can become an exact science [5,11]. Because of all his contrarian behaviour he was denied his habilitation twice and it took until 1929 until he was allowed to teach at the university and then only provided he remained within the bounds of pure mathematics.

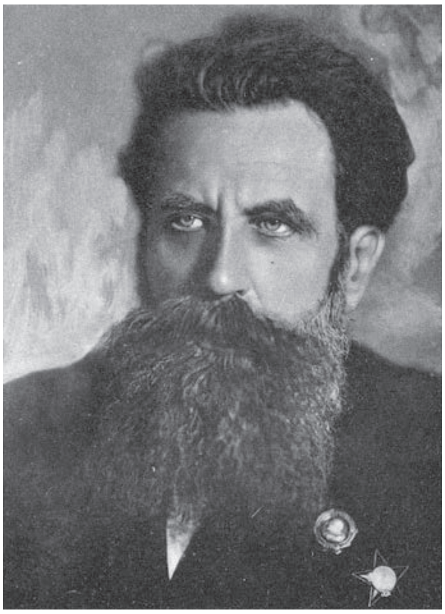
Unfortunately he did not enjoy his academic position for a long time. When the Nazis rose to power in 1933 all Jews were expelled from university. Nevertheless Remak stayed in Berlin and kept on doing mathematical research. On Kristallnacht in 1938 he was arrested and put in a concentration camp but thanks to the efforts of his non-Jewish wife he was released with a permit to go to the Netherlands. So in April 1939 he set off to Amsterdam, on his own because his wife had filed for divorce, probably partly in order to save her own life.

Robert Remak (1888–1942) [9]

In the Dutch capital he was helped out by his compatriot Hans Freudenthal, who was a lecturer at the University of Amsterdam. Again Remak's uncompromising personality caused much troubles, not only in the lectures he attended. Freudenthal also had to sort out endless problems with angry landlords to keep him from being expelled to Germany. The invasion in 1940 made matters even worse and soon Hans Freudenthal, who was also Jewish, was dismissed from the University and had to struggle for his own survival. Remak spent his final months in an apartment on the Admiraal de Ruijterweg near the Wiegbrug, until he was arrested by the Germans and sent to Westerbork. After a few weeks he was put on the train to Auschwitz where he was murdered on the 13th of November 1942.

A decorated explorer

Less than a year after Remak had published his PhD thesis, another young mathematician came up with a simplification of his proof [14,15]. Otto Schmidt, born in 1891 in a poor family of mixed Latvian-German descent, studied at the University of Kiev, which was at that time the major center for algebra in the Russian empire [8]. Because he was a very gifted student and a hard worker, he was already given research problems in his second year of study and this resulted in three papers on group theory during his bachelor degree. He also wrote a textbook on abstract groups during his master. When he completed his studies in



Otto Schmidt (1891–1956) [8]

1916 he became a lecturer at the University but then the revolution happened.

In March 1917 the czar was forced to abdicate and half a year later the Bolsheviks staged a coup, turning the Russian empire into the first communist state. During that time Schmidt was already involved in several administrative functions in the city of Kiev and due to his revolutionary sympathies he got various official appointments such as leading functions in the People's Commissariat for Food and Higher Education, the head of the State Publishing House and editor-in-chief of the *Great Soviet Encyclopedia*.

But Schmidt had higher ambitions than just becoming a dull apparatchik in the administration. Instead of staying behind his desk in Moscow he wanted to explore the outer reaches of the newly founded Soviet Union. In 1928 he headed an expedition to the Pamirs, a high mountain range in what is now Tajikistan. The same year he was also put in charge of an icebreaker to explore Franz Joseph land, a vast archipelago in the Arctic that used to be under Austro-Hungarian control before the first world war. There he established a scientific base and two years later he ventured to an even remoter archipelago in search of a viable eastward passage through the Arctic sea. In 1932 this quest was successful when he reached Vladivostok with another icebreaker. A second trip however did not go according to plan. The ship was crushed by ice and the crew consisting of over a hundred people had to be rescued by airplanes from the ice sheet.

All these exploits turned Schmidt into a national hero and Soviet celebrity. He appeared in newspapers and magazines, on radio show and post stamps, girls even hung portraits of him in their rooms. His mathematical work did not stop when Schmidt started his expeditions. He kept on teaching courses at Moscow State University and published papers on group theory well into the thirties and forties, inspiring a new generation of mathematicians in Moscow. Apart from his mathematical work he also developed a theory on the origin of our solar system and published work on binary stars and the three body problem. He founded the Institute of Theoretical Geophysics and became the vice-president of the USSR Academy of Sciences.

One might think that a man that held so many and such high office in Stalin's communist regime must have been a close ally of the dictator, but the opposite is true. In fact Schmidt was one of the only critics of Stalin's malgovernance that survived the purges in the thirties. It has been suggested that his appointment to head the first expedition to the arctic was a convenient ploy by the Soviet leader to get rid of his opponent by letting him freeze to death. When Schmidt returned and also survived the follow-up expeditions, his celebrity status probably gave him some protection from the whims of the dictator. Nevertheless in 1942 he was stripped of his vice-presidency of the Academy, but by that time his health was also deteriorating, which spared him from further political punishment.

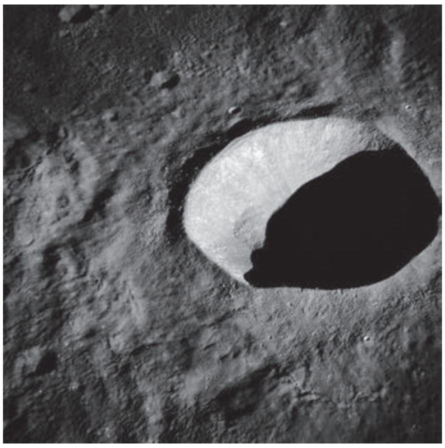


Photo: https://en.wikipedia.org/wiki/Schmidt_(lunar_crater)

Schmidt Crater, named in honor of Otto Schmidt and a bunch of other Schmidts

Otto Schmidt survived Stalin for just over three years and died of tuberculosis in 1956, an illness that had troubled him since his childhood. He is remembered as one of the most celebrated scientists of the Soviet union. He was decorated with three Orders of Lenin, two Orders of the Red Banner, an Order of the Red Star and several geographical and astronomical things are named after him: an island and a cape in the arctic, a peak in the Pamir, a crater on the Moon and even an asteroid in the asteroid belt. A truly stellar accomplishment.

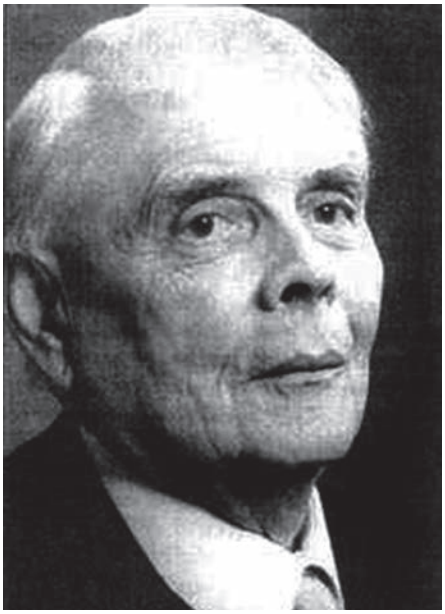
An apolitical party member

Wolfgang Krull was born in Baden-Baden in 1899 as the son of a dentist [7]. He studied at the University of Freiburg and Rostock before moving to Göttingen in 1920. At that time Göttingen was the capital of the mathematical world and the home of many famous mathematicians such a David Hilbert, Felix Klein and Emmy Noether. These last two would prove very influential for Krull's later career. Noether had been recruited by Klein and Hilbert to work on the theory of invariants and its relation to Einstein's theory of relativity but had moved on to algebra, where she became the founding mother of modern abstract algebra. Crucial to this new research area was the axiomatic approach. Instead of proving results for well-known number systems such as the real or complex numbers, quaternions or polynomials, it aims to single out the most natural or powerful property an algebraic system should satisfy in order for certain desirable theorems to hold [2].

Noether was an expert in this and the property she is still famed for is the *ascending chain condition* or ACC for short. In an object that satisfies this condition, every increasing chain of subobjects becomes stationary.

$$S_1 \subset S_2 \subset S_3 \subset \cdots \Rightarrow \exists r: \forall i,j > r: S_i = S_j.$$

In the case of abelian groups, $\mathbb{Z}$ satisfies ACC because every subgroup is of the form $k\mathbb{Z}$ and $k\mathbb{Z} \subset l\mathbb{Z}$ if and only if $l \mid k$. Therefore the ascending chain condition is equivalent to the fact that every number k has only a finite number of divisors. The additive group $\mathbb{Q}$ on the other hand does not satisfy the ascending chain condition because it contains the subgroups $2^i\mathbb{Z}$ for all $i \in \mathbb{Z}$. The ascending chain condition also has a sibling going in the opposite direction: the DCC or descending chain condition. Note



Wolfgang Krull (1899–1971) [7]

that $\mathbb{Z}$ does not satisfy the descending chain condition because we can find a decreasing set of subgroups of the form $2^i\mathbb{Z}$ with $i \in \mathbb{N}$.

Rings that satisfy the ascending chain condition for ideals are still called Noetherian rings and they share a lot of nice properties. Even today they are still heavily studied and form a basic ingredient of modern algebra and algebraic geometry, because you can squeeze a lot of juice out of this simple property. Algebraists sometimes parody Tolstoy's opening line from *Anna Karenina*:

> "All Noetherian rings are alike, each nonnoetherian ring is nonnoetherian in its own way." [12]

Although Krull was not formally a student of Emmy Noether, he was heavily influenced by her and chains of ideals feature heavily in his work. His most famous contribution, the Krull dimension of a ring measures how long chains of prime ideals in a ring can get. Krull also used the ascending and descending chain conditions to offer a new perspective on the unique decomposition theorem [3]. Instead of working with finite groups, as Remak did, Krull focused on generalized abelian groups. These are abelian groups with extra linear operators attached to and they are close cousins of what we would nowadays call modules. Krull showed that they also have a unique decomposition theorem if they satisfy both the ACC and DCC. This unifies a lot of unique decomposition theorems, such as the ones for representations of groups, algebras and Lie algebras and the Jordan decomposition of matrices.

After a brief spell in Göttingen, Krull returned to Freiburg where he obtained his PhD and became a professor in 1926. After two years he moved to the University of Erlangen where he had his most productive years, working on the interplay between commutative algebra and algebraic geometry. Among his students and collaborators Krull is known a meticulous worker who could be quite strict and demanding, but always with a human touch [6,16]. He highly valued the intuition and the aesthetics behind mathematics. His aim was not only to present mathematically correct proofs but also to present the subject in such a way that the theorems seem self-evidently true. He was admired by his students as a cultured man with a broad knowledge outside mathematics but without much interest into the social life. He was also reputed to be uninterested in politics, which makes it all the more curious that Krull had a darker side.

Krull was an official member of the Nazi party. Already in 1933 he had joined the National Socialist Teachers League and when Otto Toeplitz was dismissed from the University of Bonn because of his Jewish background, Krull was asked to take over his professorship and even became the dean of the mathematics and natural science faculty [13]. During the war he worked in the Marine observatory in Greifswald on problems relating with the weather, such as the speed of sound in the air, moisture in the air and artificial fog clouds. Near the end of the war he was taken prisoner and when he returned to Bonn he was briefly put on a list of politically incorrect professors because of his NSDAP membership. However, soon after he could take up his professorship again and he stayed in Bonn for the rest of his career until he death in 1971. We will probably never know Krull's motivations and thoughts during this episode in his life, as he was never been very vocal about it during the time or afterwards. Furthermore this aspect is barely touched upon in memorials and biographies and there is little material evidence available apart from his name appearing in membership lists.

With the rise of Nazism and communism and two world wars, the times in which our main protagonist lived were truly extraordinary and dangerous. Their fates were determined by their family roots, the places where they grew up, and their personalities, but today almost a century later these three different characters with three different backgrounds and three different lives, remain tied together by one theorem.

References

1 H. Behnke, *Semesterberichte: e. Leben an dt. Univ. im Wandel d. Zeit*, Vandenhoeck und Ruprecht, 1978.

2 L. Corry, *Modern Algebra and the Rise of Mathematical Structures*, Birkhäuser, 2012.

3 W. Krull, Über verallgemeinerte endliche abelsche Gruppen, *Mathematische Zeitschrift* 23(1) (1925), 161–196.

4 J. Maclagan-Wedderburn, On the direct product in the theory of finite groups, *The Annals of Mathematics* 10(4) (1909), 173–176.

5 U.C. Merzbach, Robert Remak and the estimation of units and regulators, in *Amphora*, Springer, 1992, pp. 481–522.

6 J. Neukirch, Erinnerungen an Wolfgang Krull, *Wolfgang Krull: Gesammelte Abhandlungen*, Walter de Gruyter, 1999, pp. 47–52.

7 J.J. O'Connor and E.F. Robertson, Biography of Wolfgang Krull, in *MacTutor History of Mathematics,* 2000, http://mathshistory.st-andrews.ac.uk/Biographies/Krull.

8 J.J. O'Connor and E.F. Robertson, Biography of Otto Yulyevich Schmidt, in *MacTutor History of Mathematics*, 2011, http://mathshistory.st-andrews.ac.uk/Biographies/Schmidt_Otto.

9 J.J. O'Connor and E.F. Robertson, Biography of Robert Remak, in *MacTutor History of Mathematics*, 2013, http://mathshistory.st-andrews.ac.uk/Biographies/Remak.

10 R. Remak, Über die zerlegung der endlichen gruppen in direkte unzerlegbare faktoren, *Journal für die reine und angewandte Mathematik*, 1911(139) (1911), 293–308.

11 R. Remak, Xxi. Kann die Volkswirtschaftslehre eine exakte Wissenschaft werden? *Jahrbücher für Nationalökonomie und Statistik*, 131(1) (1929), 703–735.

12 L. Tolstoy, *Anna Karenina*, quote adapted by Bill Crawley-Boevey in Oberwolfach.

13 F. Schmaltz, *Kampfstoff-Forschung im Nationalsozialismus: Zur Kooperation von Kaiser-Wilhelm-Instituten, Militär und Industrie,* Vol. 11, Wallstein Verlag, 2017.

14 O. Schmidt, Über die Zerlegung endlicher Gruppen in direkte unzerlegbare Faktoren, *Prot. Fiz.–Matem ob-va Pri Kiev. Univ*, 1912.

15 O. Schmidt, Sur les produits directs, *Bulletin de la Société Mathématique de France* 41 (1913), 161–164.

16 H. Schöneborn, In memoriam Wolfgang Krull, *Jahresbericht der DMV* 82 (1980), 51–62.