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History

Tic-tac geometry: A theorem in full swing

When the French mathematician Michel Chasles expressed his 'principle of correspondence' in 1864, he claimed to have obtained a theorem from which one could derive a general and systematic method for the effortless obtention of an infinity of geometrical propositions. And yet, barely two decades later, this very theorem was widely regarded as a boring or uninteresting result, whose teaching in universities might ingrain noxious mathematical habits in the minds of young students. In this article, Nicolas Michel uses the principle of correspondence as an example to show how different social contexts for the practice and teaching of science shaped different ways of valuing and measuring the worth of a result.

Transient mathematics

Recent scholarship in the history and philosophy of mathematics has done much to dismantle what may be called the 'cumulative myth'. Per this myth, it is the very nature of mathematical knowledge to monotonously increase all through history. Unlike the theories of the natural or social sciences, bound to be eventually falsified by some experiment and overtaken in dramatic revolutionary episodes, mathematical knowledge would supposedly be on a quiet upward trajectory, provided we ignore the ramblings of circle-squarers and other such pathological contributions to the perennial repository of theretofore secured theorems.

This seductive narrative, however, has now been breached from various angles. In the 1970's already, philosophers such as Imre Lakatos forcefully argued that the development of mathematical knowledge is best described as a dialectical process consisting of vague conjectures, informal proof-strategies, and nagging refutations, rather than a mere affair of axioms and undisputable inferences [17]. In parallel, many historical studies have shown that mathematical theories too can die, whether we attribute that death to sociological and external factors, scientific and internal ones, or a combination of both. One classic example is the theory of invariants studied by the sociologist Charles Fisher: a very active topic of research in the 1880's and 1890's, widely viewed as a promising prospect for the unification of various branches of mathematics, this theory had all but faded away by the 1940's. In his account of this 'death of a theory', Fischer stresses the importance of viewing mathematical edifices not just as formal constructs, but also social categories, constructed and circulated by human and institutional agents. More recently, Alma Steingart has analyzed the 'uninvention' of the proof of the Classification Theorem for finite simple groups in relation with the techniques of interpersonal communication and socialization employed by experts in this field. Further from sociological methodologies, Henk Bos has described the decline of the theory of the construction of equations (that is to say, the geometrical construction of line segments whose lengths equal the roots of an equation) in the eighteenth century as the result of the emergence of an analytical and symbolic style [4, 12, 25].

Consequently, in recent years, many historians have proposed framing the historical unfolding of mathematical knowledge as a transformative rather than cumulative phenomenon: instead of piling up theorems, collectives of mathematicians are, in this view, in the business of constantly 'rewriting' (and, sometimes, willfully misinterpreting) what their predecessors had produced. Only in the rare instances when such transformative processes cease (for instance, when the need or desire to introduce a theory into textbooks call for stable and unified presentations) can a cumulative picture of this messy historical development float back to the surface, selectively and retrospectively eliminating from its fold unsuccessful attempts, forgotten techniques, and abandoned perspectives. This view has led to new and stimulating accounts of important episodes in the history of mathematics, such as the emergence of the concept of a point on a Riemann surface, the invention of the group-concept in the wake of Galois's 1831 memoir on the resolubility of equations, or the development of Schwarz's theory of distributions [2, 11, 22].

In the following pages, I will highlight some of the insights such approaches can provide by retracing the short lifecycle of a particular geometrical theorem. Over the course of a couple of decades, the status of this theorem went from that of a general and systematic method for the effortless obtention of an infinity of geometrical propositions to a boring and uninteresting result whose unexamined teaching in universities might lead to the ingraining of bad mathematical habits. This downfall will demonstrate how different social contexts for the practice and teaching of science shaped different ways of valuing and measuring the worth of this result, as well as the importance of tacit, technical 'know-how' in the preservation or abandonment of a mathematical theory.

Chasles's principle of correspondence

The theorem in question is called the 'principle of correspondence'. First expressed in 1864 by the French geometer Michel Chasles (1793–1880), this result was the cornerstone of his 'theory of characteristics', a method for the enumeration of all conic sections in the plane that satisfy five given (independent) conditions. Though mostly forgotten today, this theory took Europe by storm in the 1860's: for it, Chasles was awarded the Copley Medal by the Royal Society; and later on it served as the basis for the development of Schubert's calculus and Hilbert's 15th problem [20, pp. 145–164, 228–233, 283–296].

Our starting point will be the notion of construction as it was conceived and mobilized by Chasles. As part of his attempt to renew the methods of projective geometry, Chasles had highlighted the importance of the concept of the cross-ratio. In particular, he had observed in 1855 that geometrical constructions that yield a one-to-one correspondence between the points of a straight line preserve cross-ratio: this he called 'the principle of anharmonic correspondence' [8]. For instance, suppose that from each point of a straight line L, we draw the two tangents to a given circle. Joining the two points of tangency, we get another straight line, which will intersect L at exactly one point. This is but one very simple case of a $(1,1)$-correspondence between the points of L; and since duality reigned supreme over all things (projectively) geometrical, Chasles formed just as many correspondences between the rays that pass through a common point I. What Chasles discovered was that, in his day-to-day geometrical practice, many such correspondences naturally arose.

Geometrical constructions, here, encompass a wide variety of techniques (such as drawing a tangent, forming an intersection, or using pole/polar constructions); the delimitation of which he never explicitly spelt out. Whilst his constructions were always based on real figures, Chasles counted as valid some points and lines which cannot be represented or drawn on an actual diagram, such as points at infinity or pairs of conjugate imaginary lines. However, no transcendental curves were allowed. Therefore, these correspondences could always theoretically be translated into algebraic equations (with real coefficients) between the abscissas of the points of the straight line, with respect to an arbitrary origin. In the case of a one-to-one correspondence between the points of abscissas x and u of the line, this equation must be of the form:

$$xu + \lambda \cdot x + \mu \cdot u + \nu = 0.$$

Indeed, this is the general form of all algebraic equations such that, when a value x (respectively u) is fixed, the resulting polynomial in u (respectively x) be linear (that is, have only one root). By expressing u as a function of x in the equation above, we find that it must in fact be a homography of the (projective) line.

Building on his projective methods to study the generation of curves via intersections of homographic pencils, Chasles would go on in the 1860's to study more elaborate correspondences, borne out of constructions which map groups of points together, and not just individual points. Let us consider another elementary example, this time based on a correspondence between rays instead of points: let C be a conic and I a point in the plane. Any ray IX passing through I will intersect C at two points, which may potentially coincide or be imaginary conjugates. At these two points, let us draw the normals to C, each of which in turn also intersect C at another point. These two resulting points U and U' define two rays IU and IU', which 'correspond' to IX. Conversely, we can use the same construction to associate two rays IX to any ray IU (see Figure 1).

This is an example of what we will call a $(2,2)$-correspondence between the rays passing through I; and we can in this fashion, using more elaborate curves and instructions, construct (α,β)-correspondences between the rays passing through a point (or, per duality, the points on a line) for any two integers α and β. Now, because such correspondences derive solely from geometrical constructions, they can also be represented by algebraic equations $P(x,u) = 0$, where x and y represent the (sine of the) angle formed by IX and IU with respect to an arbitrarily chosen direction, or in the abscissas of the points x and u (with respect to an arbitrary origin once

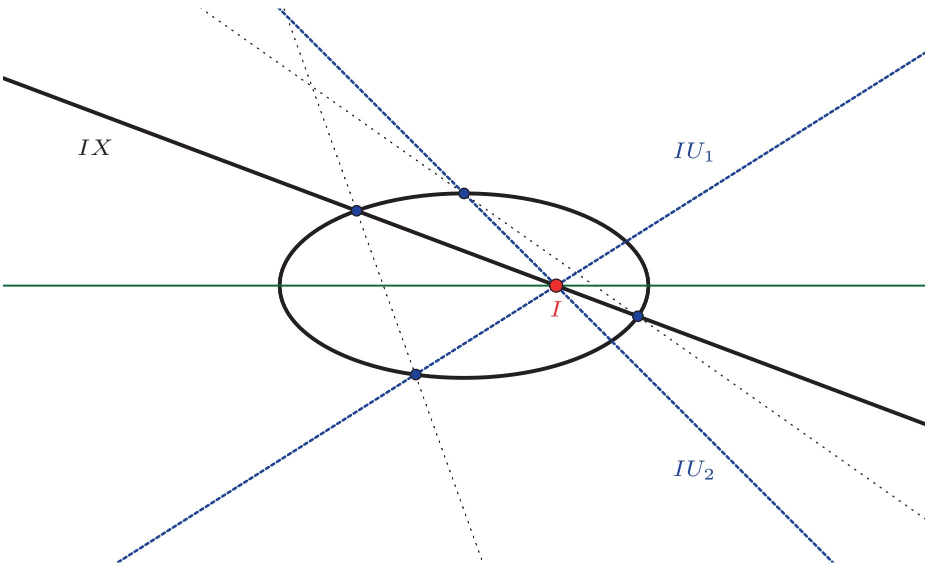


Figure 1 A $(2,2)$-correspondence.

again). However, this time, when a point x is fixed, the resulting polynomial in u will be of degree 2 (or α in the more general case); since that is the expected number of roots (and, since we allow complex roots, the Fundamental Theorem of Algebra applies). Conversely, P must be of degree 2 (respectively β) in u.

Chasles's principle gives us the number of 'coincidences' in a correspondence; that is to say, the number of points x that also belong to their own image under the correspondence. Such a point must be one of the points u which themselves correspond to x; in other words, x must be a root of the equation $P(x,x) = 0$. From elementary properties of polynomials and the Fundamental Theorem of Algebra, Chasles drew the two following (dual) results:

Theorem 1 (Chasles's Principle of Correspondence). *When, on a straight line L, two series of points x and u are such that a points u correspond to a single point x, and that b points x correspond to a single point u, then the numbers of points x which coincide with corresponding points u is $(\alpha+\beta)$.*

When, around a point I, two pencils of straight line IX and IU are such that a lines IU correspond to a single point IX, and that b lines IX correspond to a single line IU, then the numbers of lines IX which coincide with corresponding lines IU is $(\alpha+\beta)$. [9, p. 1175]

In the simple example given above, this means that our correspondence yields $(2+2) = 4$ coincidences. In fact, these coincidences turn out to be all the normals to C which can be drawn from I. The principle of correspondence has thus provided a direct answer to a geometrical problem famously discussed at length in Book V of Apollonius of Perga's *Conics*, namely that of enumerating and drawing normals to a conic from a given point [13, pp. 146–177]. In this landmark text of ancient geometry, however, separation of cases was paramount; and Apollonius had found that the numbers of normals one can draw from a point could be either 2, 3, or 4 depending on the sort of conic (parabola, hyperbola, or ellipse) and the position of the point. In Chasles's modern projective geometry, however, such distinctions were eschewed in the name of generality — one unique method provided one unique answer in one fell swoop.

A theorem thrice virtuous

At this stage, this theorem might seem rather banal and of little geometrical interest. After all, it is but the simple application of elementary algebra. Yet in the wake of its enunciation of this principle in 1864, Chasles devoted the near totality of his scientific efforts to applying it to various systems of figures (polygons, curves, harmonic axes, et cetera) until his death in 1880. Most of his writings over this period of time consist in long lists of propositions for which the proofs are rarely given, but always attributed to the principle of correspondence (see Figure 2). In his scientific archives, preserved at the Paris Académie des Sciences, one can find myriad more such propositions hastily sketched on leaflets and postcards, all of which bear witness to a prolonged and intense exploration of the potentialities of this principle.

In one of his last articles, whilst reflecting upon this theorem with which he had been so busy and productive, Chasles identified three distinct features that made it so valuable to his eyes. The first of these three virtues lies in its *simplicity*:

> "The principle of correspondence can be applied, with very great ease, to an *infinity of questions*. This ease is such that, without having to express through any equation like in Analysis the conditions of the question, *we immediately put down two numbers* which satisfy these conditions and whose simple sum expresses the solution." [10, p. 577]

In other words, one can apply the principle of correspondence to any geometrical figure, and this application is always easy: it requires no computation nor any other complicated algebraic manipulation other than the sum of two numbers.

The second virtue of this principle is that it always leads to *general* theorems:

> "By applying [this principle] to simpler questions, such as those left as exercises in the classical treatises, we immediately acknowledge that the reasoning will be absolutely the same in the case of the greatest generalization that the question can admit." [10, p. 578]

(582)

» 8. *Si, par le point où la tangente du point* a′ *rencontre une tangente fixe de* U_m, *on mène une parallèle à chaque corde* aa′, *ces parallèles enveloppent une courbe de la classe* (2m + n − 3)(m − ν).

» 9. *Si, par le point où la tangente en* a′ *rencontre une tangente fixe de* U_m, *on mène des perpendiculaires aux cordes* aa′, *ces perpendiculaires enveloppent une courbe de la classe* (2m + n − 3)(m − ν).

» 10. *Il existe* (2m + n − 2)(m − ν) *cordes* aa′ *qui sont normales à* U_m *en leur point* a′.

» *Et même nombre qui sont normales en leur point* a.

» 11. *Il existe* (m + n − 4)(m − ν) *cordes* aa′ *qui sont tangentes à* U_m *en leur point* a′.

» *Et même nombre qui sont tangentes en leur point* a.

» 12. *Du point de rencontre des tangentes en* a *et* a′ *de chaque corde* aa′ *on mène une parallèle à cette corde : ces parallèles enveloppent une courbe de la classe* 2(m + n − 2)(m − ν).

» 13. *Du point de rencontre des tangentes en* a *et* a′ *on abaisse une perpendiculaire sur la corde* aa′ : *ces perpendiculaires enveloppent une courbe de la classe* 2(m + n − 2)(m − ν).

» 14. *Les pieds des perpendiculaires sont sur une courbe de l'ordre* 2(2m + n − 2)(m − ν).

» 15. *Du point* a *de chaque corde* aa′ *on abaisse une perpendiculaire sur la droite* Oa′ : *ces perpendiculaires enveloppent une courbe de la classe* (2m − ν)(m − ν).

» 16. *Les pieds des perpendiculaires sont sur une courbe de l'ordre* (3m − 2ν)(m − ν).

Figure 2 Chasles's monotonous lists of propositions [10, p. 588].

Lastly, Chasles notes that due to the simplicity and immediateness with which it yields propositions, this principle can serve as an *art of invention*:

> "The principle of correspondences comprises such an ease of solution that, whatever the question we tackle… we immediately have the thought to apply this mode of spontaneous solution to various other questions related to the figure we have before our eyes. It is thus that, as I sought to treat some questions in the general theory of curves, I was left by the solutions to multiply by the hundreds different theorems." [10, p. 578]

An example borrowed from Poncelet

To understand how Chasles came to see these three epistemic virtues as embodied in the principle of correspondence, we need to first understand how he employed it as an instrument for the obtention of geometrical propositions. In a particularly telling passage of the same paper, Chasles considered an old theorem given by Jean-Victor Poncelet in his 1822 *Traité des Propriétés Projectives*, that is to say one of the aforementioned 'classical treatises'. Choosing Poncelet's *Traité* as the basis of this discussion was by no means a neutral choice. Chasles and Poncelet had had a fraught relationship for decades, and by showing how easily he could generalize Poncelet's results (in effect, even relegating his magnum opus to the rank of a classical but outdated treatise), Chasles was effectively staking a claim to superior mastery of the methods of pure geometry. The theorem in question is the following (see Figure 3):

Theorem 2 (Poncelet's chord theorem). *When an angle of constant magnitude α turns about its origin O, which lies on a conic $\mathcal{C}$, the chord aa' this angle intercepts on the conic envelopes another conic.* [21, p. 281]

In fact, Chasles did not prove this theorem: as we shall see, he immediately tackled a more general version thereof, using a rather idiosyncratic language and style of exposition. For the sake of clarity, however, I will first adapt his arguments to provide a more explicit and detailed proof of Poncelet's theorem using the principle of correspondence.

Proof. We show that from any fixed point I, one can draw two tangents to the envelope described in the theorem (let us call this locus $\mathcal{U}$); this is enough to show that it is a curve of class two, that is to say a conic. To that end, we consider the rays IX which turn around I and construct the following correspondence (see Figure 4).

A ray IX intersects $\mathcal{C}$ at two points a, b, to which we join O to form two straight lines Oa, Ob. We can them form the two straight lines Oa', Ob' which stand at an angle α with them, and which intersect $\mathcal{C}$ at (respectively) a' and b'. Joining I to both of these points, we have constructed two rays IU, which we associate to the initial ray IX. Conversely, to a ray IU we can also associate two rays IX via the very same construction, except with angle $-\alpha$. Per the principle, there are $2+2=4$ coincidences in this correspondence.

But of these four coincidences, two are 'foreign solutions': they do not depend on the position of I (or $\mathcal{C}$), and therefore should be discounted as mere computational artefacts. Indeed, any rotation about O fixes the same two imaginary points, namely the circular points at infinity E and F. These points, which in projective coordinates are represented by the two triplets $(1:\pm i:0)$, were then defined as the two intersections of the line at infinity and any (real) circle in the plane [21, pp. 48–49]. Thus, rotations about O map the rays OE and OF onto themselves. Now, if a is the intersection of OE and $\mathcal{C}$ (other than O), then $a=a'$; and Ia is a coincidence irrespective of where I lies. The same can be said of the intersection of OF and $\mathcal{C}$.

Consequently, there are only two 'real' coincidences; and these are, per construction, the lines aa' passing through I such that $\widehat{aOa'}=\alpha$. By definition of the locus $\mathcal{U}$, these are also all the tangents to it one can draw from I. This proves that the class of $\mathcal{U}$ is two, and $\mathcal{U}$ is indeed a conic. □

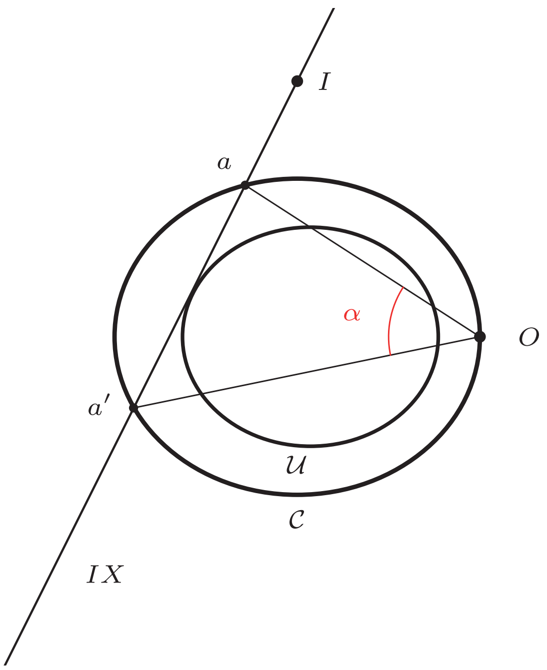


Figure 3 Poncelet's chord theorem for conics.

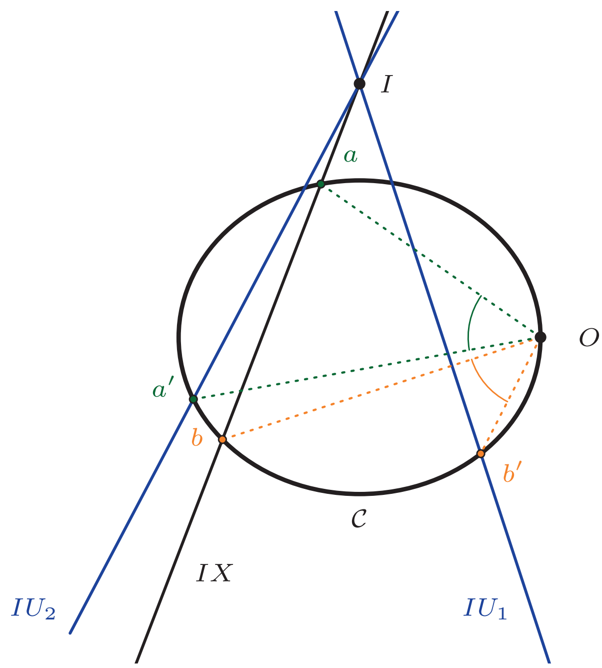


Figure 4 The $(2,2)$-correspondence used to prove Poncelet's theorem.

One may object that this proof isn't exactly as simple as Chasles had made it out to be. After all, while no computation was required, we did not merely 'put down two numbers': we had to invent a correspondence and identify foreign solutions amongst its coincidences. However, the difficulty of these tasks is largely a consequence of our lack of familiarity with such constructions. Not only does this proof (and all other proofs obtained by Chasles with the principle of correspondence) require no computations, the construction of the correspondence itself follows a systematic pattern. When setting out to establish the class of a locus, that is to say the number of tangents to it which can be drawn from an arbitrary point, Chasles always fixes a point I and forms a correspondence between the rays IX whose coincidences are the tangents to said locus. For the order of a locus, that is to say the number of points an arbitrary straight line intersects on it, he fixes a straight line L and forms a correspondence between its points x whose coincidences are the intersections of L and the locus. In both cases, the number yielded by the principle is the geometrical number being sought. And as Chasles's lists demonstrate, these two strategies can be employed to systematically and easily obtain a plethora of results. The universality and simplicity of the principle of correspondence these lists demonstrate, however, hinges on the mastery of some tacit geometrical knowledge, a technical know-how which is not encapsulated in its formal statement — and which one can only obtain by in-depth acquaintance with these texts.

To illustrate the second virtue of the principle of correspondence, Chasles goes on to explain how Poncelet's theorem can be generalized in various regards. The given conic C can be replaced by a curve U_m of order m, and the fixed point O on the curve can be of multiplicity ν (instead of being a regular point). Furthermore, the requisite that the two lines form a constant angle can be replaced by that requisite that they form a constant cross-ratio with two arbitrary, fixed lines also passing through O. Indeed, if these two fixed lines pass through the two circular points at infinity E and F, then we are left with the original condition, namely that the two mobile straight lines form a constant angle α. This is because one can define the angle between two lines L_1, L_2 (which intersect at point M) as $\frac{i}{2}\log[L_1, L_2, ME, MF]$. This result is sometimes called Laguerre's formula, and it is especially useful in computational geometry [14, pp. 147–148]. Of course, in this much more general setting, the conclusion of Poncelet's theorem must be adapted as the chords envelope a more complex curve, and Chasles gives the following statement:

Theorem 3 (Poncelet's chord theorem generalized). *When, about a point O of multiplicity n of a curve U_m, two straight lines OA, OA' turn while maintaining a constant cross-ratio m with two fixed straight lines OE, OF, the chords aa' they intercept envelope a curve whose class is* $2(m-1)(m-\nu)$. [10, pp. 579–580]

Note that, if $m=2$ and $\nu=1$, the class is $2\times 1\times 1=2$ as stipulated by the original theorem.

Chasles's key observation is that the proof of Poncelet's original theorem can be transposed in this new setting, simply substituting each term by its more general counterpart. By thus preserving the structure of the proof and simply altering its elements, one can immediately establish the more general theorem:

Proof. We fix an arbitrary point I, and we show that from I, one can draw exactly $2(m-1)(m-\nu)$ chords aa'. To that end, we construct the same correspondence as before, *mutatis mutandis*.

A ray IX intersects U_m at m points, all of which we denote a. Each point a can be joined to O by a straight line OA. The straight lines OA', which form with OA, OE, OF the required cross-ratio, all intersect U_m at $m-\nu$ points a' (because O is of multiplicity ν). In total, this procedure yields $m(m-\nu)$ points a', through which one can draw the same number of straight lines IU. Similarly, to each IU there correspond $m(m-\nu)$ straight lines IX. This is a $[m(m-\nu), m(m-\nu)]$-correspondence, which consequently yields $2m(m-\nu)$ coincidences.

However, amongst these coincidences, $2(m-\nu)$ are foreign solutions caused by the lines OE and OF. Indeed, each of these two lines intersects the curve at $m-\nu$ points a, and when IX passes through one of these points, the line OA coincides with OE and therefore with OA'. Thus, a' coincides with a, and IU with IX, wherever the point I is. There are $m-\nu$ such foreign solutions, and the same goes for the coincidences of Oa and OF. And so there are in total $2(m-\nu)$ foreign solutions, and the remaining number of 'real' solutions is:

$$2m(m-\nu)-2(m-\nu)=2(m-1)(m-\nu).$$

Lastly, these real coincidences are all the chords aa', because they are the straight lines which join two points A and A' on U_m such that OA and OA' form the required cross-ratio with OE and OF. Therefore the number of these coincidences is the class of the envelope described by the theorem. □

Using the principle (and the tacit proof-strategies associated to it) to establish a classical result, Chasles immediately obtains a proof-structure which is stable under generalization of all its terms, without the need for any further reflection. Indeed, not only is the correspondence analog to the previous one, but even the argument for the enumeration of the foreign solutions stays intact (*modulo* a simple translation). The generality of this method, for Chasles, is only worthwhile insofar as it is a systematic and effortless generality; it has little to do with the extension of the domain of application of the method, and much more to do with how this application can be carried out in a uniform manner.

Lastly, Chasles demonstrates the third virtue of the principle of correspondence by proposing another kind of alteration to Poncelet's theorem. Having generalized it to the fullest, Chasles now shows how to modify it simply by considering other properties 'of the same figure': for instance, instead of forming the locus enveloped by the chords aa', he considers the locus U' generated by the intersection of the two tangents to the curve at the points a and a' (see Figure 5, for the case where U_m is a conic C), for which he gives the following result:

Theorem 4 (Dual theorem). *U' is a curve of order $2(n-1)(m-\nu)$, where n is the class of the given curve U_m.* [10, p. 581]

This theorem is a sort of dual version of the previous one, and thus a proof comes equally easily to Chasles via the construction of a dual correspondence. This time, I give Chasles's proof with his peculiar notations and style of presentation:

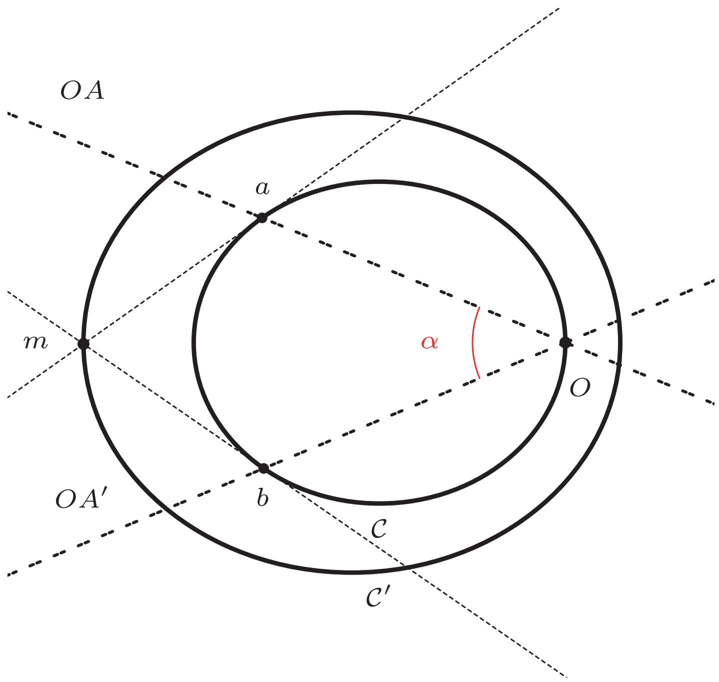


Figure 5 A dual theorem.

Proof.

$$x,\quad n\,a,\quad n(m-\nu)\,a',\,u,$$
$$u,\quad n\,a',\quad n(m-\nu)\,a,\,x,$$
$$2n(m-\nu).$$

That is to say: From a point x on a straight line L, we draw n tangents xa to U_m; to the straight lines OA which pass through these points there correspond n straight lines OA' which intersect U_m at $n(m-\nu)$ points a'; the tangents at these points intersect L at $n(m-\nu)$ points u. Similarly, a point u gives rise to $n(m-\nu)$ points x. Thus, there are $2n(m-\nu)$ points x coinciding to a corresponding point u.

But there are $2(m-\nu)$ coincidences which are foreign solutions; they are caused by the $2(m-\nu)$ intersections of the two straight lines OE and OF with U_m; since, for such a point, a' coincides with a and x with u wherever the straight line L is; it is therefore a foreign solution. There remain $2(n-1)(m-\nu)$ solutions, which is therefore the order of the curve being sought. □

Note the duality between this correspondence and the previous one: for instance, instead of considering the intersections of a straight line IX and the curve U_m, in this proof Chasles first considers the tangents to U_m drawn from a point x of a straight line L. Thus, the principle of correspondence allows for parallel alterations to the theorem and to the proofs thereof, yielding once again further geometrical truths without requiring extra effort.

Genius made accessory

Chasles's insistence on the value of simple, general, and systematic methods both as a means to prove and to discover new geometrical truths largely predated his discovery of the principle of correspondence. In 1837 already, at the very end of his celebrated *Historical survey of the development of geometrical methods*, Chasles had prophesied a very optimistic future for his science of choice in terms strikingly similar:

> "In ancient Geometry, truths were isolated. To imagine and to create new ones was a difficult task, and not everyone who so wished could become a geometer-inventor. Today, any one can come up, pick any known truth, and submit it to the various general principles of transformation; they will obtain from it other truths, different or more general; and on these ones one will be able to carry out similar operations; so that one may multiply, almost at infinity, the number of new truths deduced from the first one... Anyone who so wishes, in the current state of science, can generalize and create in Geometry; genius is no longer indispensable to add a brick to the edifice." [7, pp. 268–269]

The principle of correspondence embodied the core virtues of an ideal for mathematical practice which had motivated Chasles for decades, namely uniformity, simplicity, and generality. While the principle of correspondence was first expressed in 1864, the scientific culture underlying its use and appreciation by Chasles was in fact already a thing of the past.

As intellectual historians of early nineteenth-century France have shown, these values were those shaped and transmitted in engineering schools such as the École Polytechnique, where Chasles trained between 1812 and 1814. Created in 1794, this latter school was intended to train engineers, military men, and civil servants for the young Republic. The teaching of mathematics in the first year of its existence, of which famous *savants* such as Gaspard Monge and Joseph-Louis Lagrange were in charge, was structured around an ideal of generality: students would learn at the École Polytechnique abstract principles, and only then learn how to apply them in dedicated 'écoles d'application'. These principles, characterized by their potentially universal domain of application and by the uniformity of their march, were viewed as the most valuable and pressing lesson to promulgate amongst this young scientific elites [1; 3, pp. 114–118]. As Sylvestre-François Lacroix (1765–1843) would recount in his 1805 *Essays on teaching*:

> "The various works which a great nation must have carried out, whether it be for its defense or the improvement of its territory... require the help of almost all arts and sciences... Only very extended preparatory studies could provide the resources to direct such works. These varied studies can be sorted out in a small number of divisions which embrace a multiplicity of details too great to be gathered in a single head, but whose *general principles* are the same. To isolate these principles and to make them the subject of general instruction, ... such was the goals of the founders of the [École Polytechnique]." [16, pp. 32–33]

Thus, a suitable teaching of geometry (as of any other sciences), for the *savants* of this milieu, ought to rely primarily on the exposition of these general principles, and then on that of the art of the systematic application thereof to any potential system of figures — a framework explicitly modeled after a kind of 'engineer epistemology'. Chasles was not merely paying lip service to old professors or classmates in invoking these same values: the *tableaux* he uses in the presentation of his proofs via correspondence, the uniformity of the language, and the organization of mathematical knowledge into principles and their applications all indicate how strongly his scientific practice was shaped by this shared culture. As we shall now see, to a reader foreign to this culture, the principle of correspondence in this form would quickly lose much of its appeal.

Cosmopolitan cafés and modern mathematics

Chasles's methods initially found attentive and enthusiastic readers all over Europe: the principle of correspondence was taken up in the 1860's and 1870's by algebraic geometers Arthur Cayley and Adolf Hurwitz who expanded it to correspondences on curves of genus p instead of straight lines (in which case the number of coincidences becomes $\alpha+\beta+2kp$ for some integer k), and who used it to prove, among other things, Poncelet's famous porism (or 'closure theorem') [6; 19, pp. 264–265]. In Italy, Luigi Cremona contributed a great deal to the circulation of these methods in his teaching of geometry in Bologna. By the 1880's, the principle of correspondence had become such an integral part of advanced mathematics curricula that some even began complaining about its omnipresence. Giovanni Guccia, for instance, privately lamented in 1888 that Italian geometers wrongly focused on "endless and monotonous applications of Chasles's principle of correspondence, which [he] couldn't bear anymore" [5, p. 36]. As it turns out, Guccia was not alone in finding much to fault with the widespread use and teaching of Chasles's principle.

Corrado Segre (1863–1924) is a towering figure in the history of Italian mathematics, and his role in the shaping of modern algebraic geometry cannot be understated. After brilliant studies at the University of Turin, he was immediately hired by this institution to assist with the teaching of descriptive geometry. In 1888, he was made professor of higher geometry, a position he held for the rest of his career. Besides his own scientific contributions, Segre was an early and active promoter of Felix Klein's *Erlanger Programm*, and he trained many of the most successful Italian mathematicians of the early twentieth century, such as Gino Fano, Beppo Levi, or Francesco Severi. As historians have recently shown, Segre elected at an early stage to divert the bulk of his efforts away from publishing original research. Instead, he "devoted a large part of his time and activity to pushing his direct and distance disciples to produce original researches, which in a sense he ended up considering 'as his own'" [18, p. 96].

Mathematics, in Segre's circle, was a first and foremost a social affair. He maintained a large network of international friends, students, and colleagues, whom he often gathered in Turin. Among these, let us mention Felix Klein and Leopold Kronecker from Germany, William and Grace Young from England, René Baire and Gaston Darboux from France, and C. L. E. Moore and Ernest Wilczynski from the United States. Far from the vertical model that ruled Chasles's conception of scientific knowledge, with elite *savants* producing general principles for other to apply effortlessly but mechanically, Segre sought to foster global collaboration and the expression of individual creativity. The school of geometry which Segre organized was "linked to a very precise local milieu, constituted by the University of Turin... [but also of] cultural cafes like *Giaccardi*, *Bergia*, and the *American Bar*...In his 'little studio' various disciples and colleagues were entertained, both Italians and foreigners." Modern mathematics, for the Italian *Maestro*, was born out "conversation, and generally to all those vectors of scientific sociability... that were set alongside institutional education and, allowing greater freedom of expression and debate between the interlocutors, proved particularly useful in the creative phase of research activity" [18, pp. 117–118]. In this new locale, and under this new conception of mathematical teaching and research, Chasles's principle lost its very *raison d'être*.

Against facility

Segre's disapproval of Chasles's mathematics is at its clearest in an address to his students, which he published in 1891 in the newly-created *Rivista di Matematica* (then edited by none other than Giuseppe Peano), and which was translated into English by John Wesley Young for the *Bulletin of the American Mathematical Society* in 1904, with Segre's approval and supervision [23, 24]. Surprisingly enough, this address opens on a lengthy quote of the concluding paragraphs of Chasles's *Historical Survey*, that is to say Chasles's aforementioned diagnosis on the superfluity of genius in modern geometry. Segre initially lists, in what may seem like an endorsement of this diagnosis, some of the most important advances made by geometers over the half-century that had elapsed since the publication of Chasles's text. In many of these advances, he points to the importance of the concept of transformation, which he concedes Chasles had rightly highlighted. However, Segre is quick to correct Chasles's enthusiasm for genius-free, methodical geometry:

> "But facility is a bad counsellor; and often the work to which it leads the beginner, while it may serve as training, as preparation for original research, will not deserve to see the light... Geometric writings are not rare in which one would seek in vain for an idea at all novel... one finds [instead] *applications of known methods* which have already been made *thousands of times*; or *generalizations* from known results which are so *easily* made that *the knowledge of the latter suffices to give at once the former*, et cetera. Now such work is not merely useless; it is actually harmful because it produces a real incumbrance in the science and an embarrassment for more serious investigators; and because often it crowds out certain lines of thought which might well have deserved to be studied. Better, far better, that the student, instead of producing rapidly a long series of papers of such a nature, should work hard for a long time on *the solution of a single problem, provided it is important*: better one result fit to live than a thousand doomed to die at birth!" [24, p. 443]

The language in this passage is clear: Segre knew exactly what Chasles valued so ardently in his own research, and he outright rejected that these values held any merit for the modern student of mathematics. In the rest of his address, Segre would attempt to answer the question: "When is a question *important*?" In so doing, he outlined his own views regarding the principles a student of mathematics ought to keep in mind when selecting a topic to learn about and to research, and regarding the norms that should rule this learning and this research.

In the seventh section of this text, Segre returned with these new aims in mind to algebraic correspondences and transformations, and in particular to Chasles's own contributions to modern geometry. In his analysis of the principle of correspondence, Segre showed once again undeniable familiarity with the method of proof discussed above:

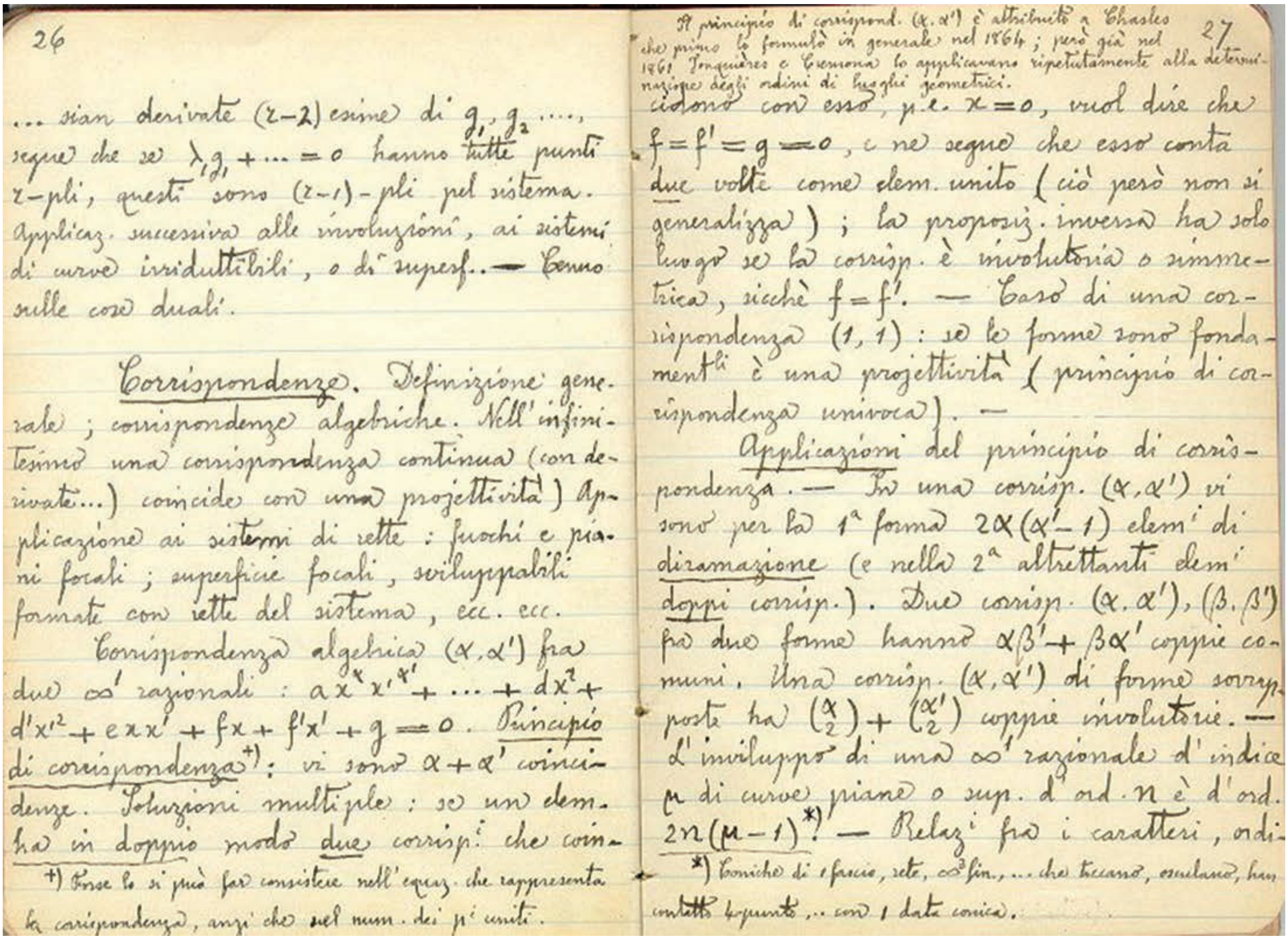
> "And so in general, just as it is easy to imagine new loci and new geometric transformations, *so also is it easy to invent applications of correspondences to obtain new truths*. We take a point A, join it to B, take the polar with regard to C, let it intersect with D, take the homologous point with regard to E, et cetera, and finally from A we obtain a point (or other element) A'; to the elements A of the first set will thus correspond the elements A' of a new set, and if A or A' is made to move in a certain way, A' or A will also move, et cetera. In this way a given figure or given property will be transformed into another, which will give rise to a third, and so on; and all *without the least difficulty*, *mechanically* as it were, *with the regularity with which a pendulum swings*."
> [In a footnote:] "Hence one of my teachers… used jokingly to call this kind of research *tic-tac-geometry*." [24, p. 458]

Here, too, Segre knew Chasles's texts in and out. His description of the generation of correspondences is faithful to Chasles's practice, and so is his assertion regarding the simplicity and systematicity of this practice. But here again, Segre completely rejects any mathematical value to this methodical and simple pendulum-like geometry: "it is not with this sort of research that we should be engaged nowadays", he asserts. What, then, should one do with the principle of correspondence, which Segre nonetheless included in his address and his teaching?

Segre's *Quaderni*: the principle's new clothes

To answer that question, one must look beyond this sole address. Fortunately, this is not the only extant trace of Segre's influential teaching. Researchers at the University of Turin have digitized some 40 notebooks (*Quaderni*) that Segre used to prepare and deliver his lectures, many of which dealt with algebraic geometry and, in particular, the principle of correspondence (see Figure 6) [15].

Correspondences figured prominently in these lectures, alongside a statement of Chasles's principle only marginally different from its 1864 antecedent (except for its generalization to correspondences on curves of genus p, which Chasles was also aware of). It would be easy to point to the many ways in which Segre was able to



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... sian derivate $(z-2)$ esime di $g_1, g_2, \dots$, segue che se $\lambda_1 g_1 + \dots = 0$ hanno tutte punti z-pli, questi sono $(z-1)$-pli pel sistema. Applicaz. successiva alle involuzioni, ai sistemi di curve irriduttibili, o di superf. — Cenno sulle cose duali.

Corrispondenze. Definizione generale; corrispondenze algebriche. Nell'infinitesimo una corrispondenza continua (con derivate...) coincide con una projettività) Applicazione ai sistemi di rette: fuochi e piani focali; superficie focali, sviluppabili formate con rette del sistema, ecc. ecc.

Corrispondenza algebrica (α, α') fra due ∞' razionali: $a x^{\alpha} x'^{\alpha'} + \dots + d x^2 + d' x'^2 + e x x' + f x + f' x' + g = 0$. Principio di corrispondenza+): vi sono $\alpha + \alpha'$ coincidenze. Soluzioni multiple: se un elem. ha in doppio modo due corrisp.i che coin-

+) Forse lo si può far consistere nell'equaz. che rappresenta la corrispondenza, anzi che nel num. dei p.ti uniti.

Il principio di corrispond. (α, α') è attribuito a Chasles che primo lo formulò in generale nel 1864; però già nel 1861 Jonquières e Cremona lo applicavano ripetutamente alla determinazione degli ordini di luoghi geometrici.

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cidono con esso, p.e. $x = 0$, vuol dire che $f = f' = g = 0$, e ne segue che esso conta due volte come elem. unito (ciò però non si generalizza); la proposiz. inversa ha solo luogo se la corrisp. è involutoria o simmetrica, sicchè $f = f'$. — Caso di una corrispondenza $(1, 1)$: se le forme sono fondamentli è una projettività (principio di corrispondenza univoca). —

Applicazioni del principio di corrispondenza. — In una corrisp. (α, α') vi sono per la 1a forma $2\alpha(\alpha'-1)$ elem.i di diramazione (e nella 2a altrettanti elem.i doppi corrisp.). Due corrisp. (α, α'), (β, β') fra due forme hanno $\alpha\beta' + \beta\alpha'$ coppie comuni. Una corrisp. (α, α') di forme sovrapposte ha $\binom{\alpha}{2} + \binom{\alpha'}{2}$ coppie involutorie. — L'inviluppo di una ∞' razionale d'indice μ di curve piane o sup. d'ord. n è d'ord. $2n(\mu-1)$*)? — Relaz.i fra i caratteri, ordi-

*) Coniche di 1 fascio, rete, ∞^3 fin., ... che toccano, osculano, hanno contatto 4punto, ... con 1 data conica.

Figure 6 Fondo Corrado Segre, Library of the University of Turin, *Quaderni* 2 (1889–1890), pp. 26–27. Available at www.corradosegre.unito.it/quaderni.php.

apply the principle of correspondence to many new objects which Chasles could not have studied, such as surfaces in n-dimensional or non-Euclidean spaces. The real change, however, appears in the sorts of tasks which Segre expected his students to carry out with this principle.

In his courses on algebraic geometry, the very first applications of the principle of correspondence are always concerned with the theoretical properties of (α, α')-correspondences, for instance investigating pairs of coinciding points common to two different correspondences, or even considering the composition of several correspondences. Elsewhere, Segre described the role of correspondences to be that of the basis of the theory of birational transformations, which in turn was to provide tools for the study of higher singularities of skew curves and surfaces. While these questions might have made mathematical sense to Chasles, they were completely foreign to his interests. The principle of correspondence, for Chasles, was an instrument to transform and multiply at will geometrical statements, but in itself it was no object of research. For Segre, by contrast, this mechanical multiplication was not a desirable form of mathematical life, and so he directed his students toward the study of the general properties of this new kind of transformation.

It is obvious from a mere glance at a recent textbook on algebraic geometry that Segre's viewpoint has won the cultural battle for what mathematical life should look like, as well as the mathematical battle that for what is worth discussing in Chasles's principle of correspondence. However, one may pause to reflect on the knowledge that has been lost in the process: Chasles's skillful constructions of correspondences, with which he has so enriched our understanding of algebraic curves and surfaces, is completely forgotten now, even to those who may master the algebraic intricacies of the notion of correspondence. Segre was still in possession of this skill; and yet, he chose not to pass it down to his students, and to foster instead the study of other avenues for increasing the knowledge of geometrical correspondences. All of this points to the fact, hard-won by historians of mathematics in recent years, that a theorem is not just a formal statement, a predicate with a domain of validity or an absolute equation. It is also a set of associated practices and techniques, many of which are tacit — and therefore, susceptible to be forgotten, transformed, or to reappear unexpectedly.

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