Problemen

| Problem Section

This Problem Section is open to everyone; everybody is encouraged to send in solutions and propose problems. Group contributions are welcome. We will select the most elegant solutions for publication. For this, solutions should be received before **15 April 2021**. The solutions of the problems in this issue will appear in the next issue.

Problem A (proposed by Daan van Gent and Hendrik Lenstra)

Let G and A be groups, where G is denoted multiplicatively and where A is abelian and denoted additively. Assume that A is 2-torsion-free, i.e. it contains no element of order 2.

Suppose that $q: G \to A$ is a map satisfying the parallelogram identity: for all $x, y \in G$ we have

$$q(xy) + q(xy^{-1}) = 2q(x) + 2q(y).$$

Prove that for all $x, y \in G$ we have $q(xyx^{-1}y^{-1}) = 0$.

Problem B (folklore)

Prove that every Jordan curve (i.e. every non-self-intersecting continuous loop in the plane) contains four points A, B, C, D such that $ABCD$ forms a rhombus.

Problem C (proposed by Daan van Gent)

A *directed binary graph* is a finite vertex set V together with maps $e_1, e_2 : V \to V$. (The edges are formed by the ordered pairs $(v, e_i(v))$ with $i \in \{1,2\}$.)

For $a,b,c,d \in \mathbb{Z}_{>0}$, an $(a:b)$*-to-*$(c:d)$ *distributive graph* is a directed binary graph G together with distinct vertices $s, t_1, t_2 \in V$ such that G interpreted as a Markov chain has the following properties:

1. For all $v \in V$ the edges $(v, e_1(v))$ have transition probability $\frac{a}{a+b}$ and edges $(v, e_2(v))$ have probability $\frac{b}{a+b}$.
2. It has the initial state s with probability 1.
3. Both t_1 and t_2 connect to themselves, meaning $e_i(t_j) = t_j$ for all $i, j \in \{1,2\}$.
4. It has a unique stationary distribution of t_1 with probability $\frac{c}{c+d}$ and t_2 with probability $\frac{d}{c+d}$.

Show that for all $a,b,c,d \in \mathbb{Z}_{>0}$ there exists an $(a:b)$-to-$(c:d)$ distributive graph.

Edition 2020-3 We received solutions from Brian Gilding, Pieter de Groen, Marco Pouw and Ludo Pulles.

Problem 2020-3/A (proposed by Onno Berrevoets)

Let $f : (-1,1) \to \mathbb{R}$ be a function of class C^∞, i.e., all higher derivatives of f exist on $(-1,1)$. Let $c \geq 0$ be a real number. Suppose that for all $x \in (-1,1)$ and all $n \in \mathbb{Z}_{\geq 0}$ we have $f^{(n)}(x) \geq -c$. Also assume that for all $x \in (-1,0]$ we have $f(x) = 0$. Prove that f is the zero function.

Solution We received solutions from Brian Gilding, Pieter de Groen and Marco Pouw. This solution is based on the one by Brian Gilding, who not only gives a very concise solution, but also shows that some of the assumptions can be weakened.

Since $f \in C^\infty(-1,1)$ and $f \equiv 0$ in $(-1,0]$, $f^{(n)}(0) = 0$ for every $n \in \mathbb{Z}_{\geq 0}$. Consequently, for arbitrary $x \in (0,1)$ and $n \in \mathbb{Z}_{\geq 2}$, Taylor's Theorem (or repeated integration by parts, following the proof by Pieter de Groen) gives

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$$f(x) = \int_0^x \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t)dt \geq -c\int_0^x \frac{(x-t)^{n-1}}{(n-1)!}dt = -c\frac{x^n}{n!}.$$

Likewise,

$$f'(x) = \int_0^x \frac{(x-t)^{n-2}}{(n-2)!} f^{(n)}(t)dt,$$

and this gives us

$$f(x) - \frac{x}{n-1}f'(x) = -\int_0^x \frac{t(x-t)^{n-2}}{(n-1)!} f^{(n)}(t)dt \geq c\int_0^x \frac{t(x-t)^{n-2}}{(n-1)!}dt = \frac{c}{n-1}\frac{x^n}{n!}.$$

Passage to the limit $n \to \infty$ yields $f(x) = 0$ for all such x.

The assumption $f^{(n)} \geq -c$ in $(-1,1)$ for every $n \in \mathbb{Z}_{\geq 0}$ for some nonnegative real number c can be relaxed to $\pm f^{(n)} \leq n! g_n$ for every $n \in \mathbb{Z}_{\geq 0}$ for a sequence of nonnegative functions $\{g_n : n \in \mathbb{Z}_{\geq 0}\} \subset L^\infty_{\mathrm{loc}}(-1,1)$ with the property $x^n \| g_n \|_{L^\infty(-x,x)} \to 0$ as $n \to \infty$ for all $x \in (0,1)$. Furthermore, given that $f^{(n)}(0) = 0$ for every $n \in \mathbb{Z}_{\geq 0}$, it is not necessary to suppose that $f \equiv 0$ in $(-1,0)$. This can be shown analogously to $f \equiv 0$ in $(0,1)$.

Problem 2020-3B (proposed by Onno Berrevoets)

Consider the map $f : \mathbb{Z}^2_{\geq 0} \to \mathbb{Z}^2_{\geq 0}$, $(a,b) \mapsto (2\min\{a,b\}, \max\{a,b\} - \min\{a,b\})$.
We call $(a,b) \in \mathbb{Z}^2_{\geq 0}$ *equipotent* if there exists $n \in \mathbb{Z}_{\geq 0}$ such that $f^n(a,b) = (x,x)$ for some $x \in \mathbb{Z}_{\geq 0}$ (where $f^n = f \circ \cdots \circ f$). Show that $(a,b) \in \mathbb{Z}^2_{\geq 1}$ is equipotent if and only if $\frac{a+b}{\gcd(a,b)}$ is a power of 2.

Solution We received solutions by Pieter de Groen and Ludo Pulles. This solution is based on the one by Pieter.

It is clear that $f(ca,cb) = cf(a,b)$ for all non-negative integers a, b, k, and it follows that (ca,cb) is equipotent if and only if (a,b) is. So it suffices to show that for $a,b \geq 1$ relatively prime, we have (a,b) is equipotent if and only if $a+b = 2^k$ for some $k \geq 1$.

$\Leftarrow$: Suppose that $a,b \in \mathbb{Z}_{\geq 1}$ are relatively prime and satisfy $a+b = 2^k$. If $k=1$, we have $(a,b) = (1,1) = f^0(1,1)$ is equipotent. If $k > 1$ and $(c,d) := f(a,b)$, then $c = 2\min(a,b)$ is even, and hence so is d because $c+d = a+b$ is even. Hence (a,b) is equipotent with sum 2^k if and only if $(\frac{c}{2}, \frac{d}{2})$ is equipotent with sum 2^{k-1}. Note that $\frac{c}{2}$ and $\frac{d}{2}$ are relatively prime because $\gcd(2\min\{a,b\}, \max\{a,b\} - \min\{a,b\})$ can only take on the values $\gcd(a,b)$ or $2\gcd(a,b)$. We can conclude (a,b) is equipotent by induction.

$\Rightarrow$: Conversely, suppose that $a,b \in \mathbb{Z}_{\geq}1$ are relatively prime with $a+b$ not a power of 2. Note that the sum of (a,b) is invariant under f, because if $f(a,b) = (p,q)$, we have $p+q = \max(a,b) + \min(a,b) = a+b$. If $a+b$ is odd, then the same is true for $f^n(a,b)$, so (a,b) is not equipotent. Suppose $a+b$ is even. Because a,b are relatively prime, both a and b are odd. Now similar to the above, if $f(a,b) = (c,d)$, then both c and d are even, and (a,b) is equipotent if and only if $(\frac{c}{2}, \frac{d}{2})$ is. Since $\frac{c}{2} + \frac{d}{2} = \frac{a+b}{2}$ and $\frac{c}{2}, \frac{d}{2}$ are relatively prime, repeating this procedure eventually results in a pair with odd element-sum, which is not equipotent. Hence (a,b) was not equipotent either.

Problem 2020-3/C* (folklore)

Uncle Donald cuts a 3 kg piece of cheese in an arbitrary, finite number of pieces of arbitrary weights. He distributes them uniformly randomly among his nephews Huey, Dewey and Louie. Prove or disprove: the probability that two of the nephews each get strictly more than 1 kg is at most two thirds.

Solution This problem remains open. This is a Star Problem for which the proposer does not know any solution. For the first correct solution sent in within one year there is a prize of €100.