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Event Abel Prize 2020

Life and work of Hillel Furstenberg

The Abel Prize 2020 was awarded to Hillel Furstenberg and Gregory Margulis "for pioneering the use of methods from probability and dynamics in group theory, number theory and combinatorics" (from the press release). Benjamin Weiss has worked closely with Hillel Furstenberg for more than fifty years. He describes Furstenberg's life and work, highlighting some of his major contributions to topological dynamics and ergodic theory.

Hillel Furstenberg was born in 1935 in Berlin. His family succeeded in escaping from Nazi Germany soon after the Kritallnacht pogrom in November 1938. Their first stop was in England where his father died and with his mother and older sister they came to the United States in 1940. For the first year they lived with an uncle who had a poultry farm in New Jersey. They then moved to the Washington Heights neighborhood in New York City and his high school and college studies were at the Yeshiva University complex. While still in high school his mathematical abilities became evident and he was greatly encouraged and supported by Jekuthiel Ginsburg, the founding editor of the *Scripta Mathematica*. This was one of the very few journals at that time dedicated inter alia to the history of mathematics and its recreational aspects. Jekuthiel Ginsburg employed him in various tasks connected with the journal including translating articles from French and German.

He graduated Yeshiva University in 1955 with two degrees, both BA and MA and went on to Princeton University for his doctoral studies. His advisor was Salomon Bochner who just in that year, 1955, had published his influential book *Harmonic Analysis and the Theory of Probability*. Hillel submitted his thesis titled simply *Prediction Theory* in 1958. In the same year he married Rochelle Cohen from Chicago who had been very active in Bnei Akiva, a religious Zionist youth movement. During the following year he was an instructor at Princeton and then at MIT. In 1961 became an assistant professor at the University of Minnesota. He was a visiting professor at Princeton in 1963–1964 where I was privileged to take a course in probability theory with him. The last lectures of this course were devoted to a new concept 'absolute independence' which evolved into the theory of disjointness (see below). When he returned to Minnesota he was granted tenure as a full professor.

In 1965 he accepted an offer from the Hebrew University of Jerusalem, which he accepted, and it has remained his academic home until today. He supervised many students for their PhD degree, many of them became quite prominent. Most have remained in Israel and he is to a large extent responsible for Israel's becoming a leading center in ergodic theory, topological dynamics and especially their applications to group theory, number theory and combinatorics. Prior to the Abel Prize he has received many prizes for his work including the Israel Prize and the Harvey Prize (1993), the Emet Prize (2004) and the Wolf Prize (2007).

1950s

Rather than trying to divide his work into different areas I will describe his work and its vast impact in a more or less chronological order. There is a commonly accepted wisdom that every colloquium talk in mathematical should contain at least one proof. In a similar vein I will include in this survey a complete summary of one of the first papers that Hillel wrote while he was still an undergraduate. It is titled 'On the infinitude of primes' and appeared in the *American Mathematical Monthly* [7]. (Hillel's official first name is Harry and all of his earlier papers were published under that name. His Hebrew name is Hillel and he has been using it for all of his publications in the last forty years.) In it he gives a highly original proof of Euclid's theorem as follows. He topologizes the integers, positive as well as negative, by taking for a base for the topology all arithmetic progressions. In this topology each arithmetic progression is both closed and open. Now by contradiction assume that the set of primes $\{p\}$ is finite. Denote by A_p all multiples of p. These are arithmetic progressions and their union is therefore closed. But the complement consists of $\{-1,1\}$ which is clearly not open, hence we have arrived at a contradiction.

1960s

His first major work was an expanded version of his thesis, published in 1960 in the *Princeton Annals of Mathematical Studies* under the title 'Stationary processes and prediction theory'. While at that time the prediction theory of stochastic processes was already highly developed it was Hillel's idea to study the possibility of developing a theory which involves only a single sample sequence. To accomplish this he used the pointwise ergodic theorem to show how to recover the distribution of an ergodic stationary stochastic process from a single typical sample sequence. He then introduced new methods from harmonic analysis to prove some special cases of the prediction problem. He also obtained as a byproduct of these new methods, dynamical proofs of some of the classical theorems of Hermann Weyl on equidistribution. Many of the original ideas that pervade his later work appear already in this monograph, applied to a very specific problem in probability.

In that same year (1960) he published, jointly with Harry Kesten, a fundamental paper [14] on the limiting behavior of the products of a sequence of independent, identically distributed random matrices. Their results were motivated in part by an observation made by Marc Kac that solving certain second order differential equations sometimes entails studying random products of 2×2-matrices. This paper laid the foundations for the modern study of products of random matrices which has turned out to have many applications in mathematics, physics and computer science. It also initiated Hillel's interest in the general study of non-commuting random products and the structures needed to describe their limiting behavior.

The results of these investigations were published in 1963, Hillel's annus mirabilis. Gian-Carlo Rota began his review of this paper [9] in the *AMS Math Reviews* with the words "This is a profound memoir", and instead of attempting to summarize it quoted at length Hillel's clear introduction. It is here that the fundamental role of stationary measures is exploited for the first time. This was also one of the motivations Hillel had for embarking on a highly original study of Lie Groups and their actions to which I return later. Before explaining what stationary measures are I will need to explain some of the basic concepts of topological dynamics which are perhaps not as well known as probability theory.

The abstract discipline had its origins in the qualitative study of solutions to the differential equations of classical mechanics. As such it was mainly concerned with actions of the reals on Euclidean space and their asymptotic properties. In the more abstract approach one takes as basic objects pairs (X,G,α) where X is a topological space, often assumed to be compact, and G is a topological group. The α is a continuous mapping from $G\times X$ to X, which for fixed $g\in G$ is a homeomorphism of X and as functions from X to X, $\alpha(g,\cdot)$ is a homomorphism from G to the group of homeomorphisms of X. The classic examples have $G=\mathbb{R}$, while the simplest case is when $G=\mathbb{Z}$ and then we often write simply (X,T) where T is a homeomorphism and corresponds to 1, so that one wants to study the properties of T^n, especially as n tends to infinity. We often abbreviate $\alpha(g,x)$ to simply gx where it is understood that we are discussing a fixed action. The *orbit* of a point x is the set $\{gx\}$ as g ranges over all the elements of the group.

Such an action is said to be *topologically transitive* if there is at least one dense orbit, i.e. at least one point such that the closure of its orbit is X. An action is said to be *minimal* if all orbits are dense. Closed G-invariant subsets E of X are called *minimal sets* if the restriction of the action to E is minimal. Any action contains topologically transitive subsets, and if X is compact then there is always at least one minimal set. This latter fact follows easily by an application of Zorn's lemma. Perhaps the simplest examples of minimal actions of $\mathbb{Z}$ are irrational rotations of the circle, or more generally the higher dimensional tori $\mathbf{T}^d$.

One of the basic tools in topological dynamics is that of invariant measures. For groups G like $\mathbb{R}$, $\mathbb{Z}$, and more generally for all amenable groups, every action of G on a compact space has at least one invariant measure. For non-amenable groups like semi-simple non-compact groups invariant measures may not exist. However, for any probability measure μ on G there always exist measures ν on X such that $\int g*\nu d\mu(g)=\nu$ where $g*\nu$ is the push forward of the measure ν under the homeomorphism of X that is given by $g\in G$. Such measures are called *stationary measures*.

Hillel's proof of Weyl's equidistribution theorem was based on showing that certain kinds of extensions of these toral rotations have a property which is even stronger than minimality which is called *strict ergodicity*. This means that the action has a unique invariant measure with global support. These extensions were homeomorphisms of $\mathbf{T}^{d+r}$ of the form $F(u,v)=(u+\theta,v+f(u))$ where θ is an irrational rotation of $\mathbf{T}^d$ and f is a continuous function from $\mathbf{T}^d$ to $\mathbf{T}^r$. These actions are called *skew-products* because the second coordinate is being acted upon by a variable homeomorphism. In his thesis he took f to be given by some special integer matrix, and in the paper [8] he took up the general question as to when is such an extension, and even a tower of such extensions, strictly ergodic. This led him to the study of general *distal* actions. These are actions with the property that for all pairs of distinct elements $(x,y)\in X\times X$ the closure of the orbit $\{(gx,gy):g\in G\}$ does not intersect the diagonal. Since the second coordinate is being acted upon by isometries the functions F of the above form define distal actions of $\mathbb{Z}$.

The simplest kinds of actions are the *equicontinuous* ones in which the homeomorphisms $\alpha(g,\cdot)$ form an equicontinuous family. In this case their closure in the group of homeomorphisms with the uniform topology form a compact group K and the space decomposes into minimal sets where each minimal set is a homogeneous space of K and the action by G is simply given by multiplication. The irrational rotations are just such examples. Naturally such actions are distal.

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Hillel Furstenberg

In [10] Hillel gave a beautiful concrete description of the structure of an arbitrary distal action of any locally compact group G. This structure theorem is the cornerstone of all later structure theorems in topological dynamics and served as the model for the structure theorems in ergodic theory which played such a key role in his remarkable ergodic theoretic proof of Szemeredi's theorem to which I will return below. In order to formulate the result we need some more definitions. If we have two actions α, β of G on compact spaces X, Y and a continuous mapping π from X onto Y that is equivariant, i.e. $\beta(g, \pi(x)) = \pi(\alpha(g,x))$ then we say that the action α is an *extension* of β while β is called a factor of α. In the case when the acting group is $\mathbb{Z}$ if the two systems are $(X,T),(Y,S)$ the equivariance is simply $\pi T = S\pi$.

Now the main notion we need is that of an *isometric extension*. An extension of an action on Y which is given by the map $\pi : X \to Y$ is called an *isometric extension* of Y if all fibers $\pi^{-1}(y)$ carry the structure of a homogeneous space K/L of a compact metric group K and there is a continuous metric $d(x_1,x_2)$ defined on all pairs (x_1,x_2) such that $\pi(x_1) = \pi(x_2)$ such that the fibers are isometric to K/L and finally the key property: $d(x_1,x_2) = d(gx_1,gx_2)$ for all such pairs and all $g \in G$. The simplest such situation is when $X = Y \times K/H$ and for a continuous function $f : Y \to K$ one has $T(y,kH) = (Sy, f(y)kH)$. It is straightforward to check that an isometric extension of a distal system is also distal. If one has an infinite tower of successive extensions one readily obtains an action on the inverse limit of the compact spaces which is of course compact. Such an inverse limit of isometric extensions is also distal. The remarkable structure theorem that Hillel proved states:

Theorem 1. *Every distal system can be obtained from the trivial one point system by a combination of the operations of forming an isometric extension and taking inverse limits.*

In the first step of the tower, namely an isometric extension of the trivial system, one obtains an equicontinuous system. An immediate corollary of the theorem is then the surprising result that any distal system has an equicontinuous factor. It follows that groups with no equicontinuous actions, such as $\mathrm{SL}(d,\mathbb{R})$ have no distal actions. However even for such groups there can be isometric extensions. More general structural results require some more definitions. The opposite of an isometric extension is a proximal extension. A pair of points (x_1,x_2) is called *proximal* if the closure of $\{(\alpha(g,x_1),\alpha(g,x_2)) : g \in G\}$ in $X \times X$ has a nonempty intersection with the diagonal. An action (X,G,α) is called a *proximal action* if every pair of points (x_1,x_2) is proximal. An extension $\pi : X \to Y$ is called a *proximal extension* if every pair (x_1,x_2) such that $\pi(x_1) = \pi(x_2)$ is proximal. Later work by R. Ellis, E. Glasner, W. Veech and others showed that a very large class of actions, so called PI (proximal-isometric), can be obtained by a succession of proximal and isometric extensions beginning with the trivial action on one point. All of these results rely in a fundamental way on Hillel's structure theorem.

The third paper of that year (1963) is titled 'A Poisson formula for semi-simple Lie groups' [11]. One of the motivations for that work came from the work on distal systems and isometric extensions. When one has a 'skew product transformation': $T(x,y)=(S(x),R(x)(y))$, where S is a fixed transformation, $R(x)$ is a transformation on the y-coordinate depending on x and one iterates this transformation, one is led to product transformations of the form $R^n(x)\cdots R^3(x)R^2(x)R(x)$. Assuming the transformations $R(x)$ come from a given group of transformations, one can hope to analyze this product in terms of properties of the group. The complexity of this expression led Hillel to consider the case where the successive $R^n(x)$ are independent random variables. For the case of a matrix group this is what Hillel worked on with Kesten. Now he turned to the general case of random walks of groups.

It is in this paper that the important notion of a strongly proximal action was introduced (although the name came later). An action of G on a compact space X is called *strongly proximal* if the action induced on the probability measures on X is proximal. For minimal actions this is the same as saying that for any probability measure μ on X there is a net g_i such that $g_i * \mu$ converges in the weak^*-topology to a point mass. For a semi-simple Lie group G a compact homogeneous space $G/H=M$ is called a *boundary* if the natural action of G on M is strongly proximal. Hillel identified a unique maximal boundary which is now called the Furstenberg boundary of G. The Poisson formula referred to in the title is a generalization of the classical Poisson formula which represents every bounded harmonic function in the unit disk as an integral of a continuous function on the circle with respect to the Poisson kernel. Hillel defines harmonic functions on G and then gives an analogue of this formula using a kernel which is defined on the Furstenberg boundary. These boundaries have turned out to play an important role in areas far from the original probabilistic motivations, coming from random walks on G (a special case of this was the original paper on products of random matrices), such as the theory of C^*-algebras.

Just a few years later, in 1967, Hillel published a landmark paper titled 'Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation' [12]. I can do no better in describing the impact of this work than by quoting from the review by Bill Parry (who was at that time one of the leading figures in ergodic theory and topological dynamics):

> "The approach to ergodic theory in this remarkable paper is complementary to the one developed, mainly by the Russian school, associated with numerical and group invariants. In fact, the relationship investigated here between two measure-preserving transformations (processes) and between two continuous maps (flows) is disjointness, an extreme form of non-isomorphism. The concept seems rich enough to warrant quite a few papers, and these papers will no doubt be largely stimulated by the present one. An interesting aspect of the paper, apart from the new results it contains, is the entirely novel demonstration of a number of established theorems."

Needless to say his prediction has been more than fulfilled. I will not try to explain what disjointness is except to say that there are two separate (albeit similar) definitions, one for measure preserving systems and one for topological dynamical systems. He applied the measure theoretic version to a problem of filtering signals that have been distorted by random noise. On the topological side he proved a basic result about disjointness in topological dynamics and used this in a remarkable application of these new ideas to Diophantine approximation which I will now describe.

Among the equidistribution theorems that H. Weyl proved in his famous 1916 paper is the result that for any increasing sequence of integers a_n the fractional parts of $a_n x$ are equidistributed for a.e. real number x. For $a_n=n$ the only exceptions are rational numbers. However if a_n is a lacunary sequence, i.e. the ratios a_{n+1}/a_n are bounded away from one, then there are uncountably many exceptions. While the semigroup p^n is lacunary, the semi-group generated by co-prime integers p,q, i.e. $\{p^n q^m\}$ is easily seen to be non-lacunary. Hillel showed that for any non-lacunary semi-group Σ of integers and every irrational θ the fractional parts of $\{\sigma\theta : \sigma\in\Sigma\}$ are dense in $[0,1]$. Another way to formulate this is that the only closed invariant sets for the action of Σ on the unit circle $\mathbf{S}=\{z\in\mathbb{C}: |z|=1\}$ by $z\to z^\sigma$ are finite sets of roots of unity and $\mathbf{S}$ itself. In contrast, for the action on $\mathbf{S}$ which is generated by a single integer k, there is a host of distinct invariant closed sets. These invariant sets can have arbitrary Hausdorff dimension between zero and one. This was the first 'rigidity result' that exhibited the dramatic difference between the action of a single transformation and an action by several commuting transformations.

In the setting of ergodic theory Hillel suggested that an analogous result should be true. Namely that the only purely non-atomic measure on $\mathbf{S}$ invariant under such a semigroup Σ is Lebesgue measure. This problem remains open until today.

The best partial results were obtained more than thirty years ago in works by R. Lyons, D. Rudolph and A. Johnson [29, 30, 31]. Without entering the detailed story the final result they achieved was that if the semigroup leaves a measure μ invariant, and μ has positive entropy under the action of some element of the semigroup then μ equals Lebesgue measure.

In most of the more recent works on rigidity of higher rank actions of algebraic origin there is a similar important role played by the entropy. It is these results in homogeneous dynamics that have had many applications to number theory and Diophantine approximation not to mention quantum unique ergodicity. I will say more about these matters below.

1970s

Returning to what came after the paper on disjointness, Hillel realized that the Diophantine result suggests some sort of transversality between the actions of multiplication by p and multiplication by q. He explored this in a lecture in 1969 at a symposium honoring his mentor S. Bochner [13] where he again developed some new ideas and formulated several conjectures which have stimulated many further developments. The study of the Hausdorff dimension of sets of the type that Hillel was considering in this paper was quite a narrow field fifty years ago. The work of B. Mandelbrot on fractals popularized the field and was one of the stimuli to many attempts to settle these conjectures. Quite recently one of these conjectures was established independently by P. Shmerkin [32] and M. Wu [33] and I shall describe this in some detail as a nice illustration of the depth of Hillel's insights.

Let dim denote the Hausdorff dimension function for subsets of (X,d), a compact metric space. Two closed subsets A, B of X are defined by Hillel to be *transverse* if

$$\dim(A\cap B)=\max\{0,\dim A+\dim B-\dim X\}.$$

Two continuous mappings T,S from X to X are said to be *transverse* if for all closed sets A and B that are T and S invariant, respectively the sets A, B are transverse.

The conjecture in question is:

Conjecture (1970). *Two positive integers a,b are said to be* multiplicatively independent, *if the ratio of their logarithms is irrational. For an integer m denote by T_m the map of the unit interval to itself defined by $T_m(x)=mx\,(\mathrm{mod}\,1)$. The mappings T_a, T_b are transverse for all multiplicatively independent integers a,b.*

The two proofs of this conjecture by Shmerkin and Wu are quite different. Wu's proof makes use of one of the novel tools developed by Hillel in [13] to obtain some partial results towards his conjecture. Hillel introduced there spaces of measures on trees and Markov processes on these spaces. These ideas were later elaborated and given a geometric form as 'CP-processes'. Rather than giving a detailed definition of these processes I will quote Hillel's abstract to his 2008 paper titled 'Ergodic fractal measures and dimension conservation' [20]:

> "A linear map from one Euclidean space to another may map a compact set bijectively to a set of smaller Hausdorff dimension. For 'homogeneous' fractals (to be defined), there is a phenomenon of 'dimension conservation'. In proving this we shall introduce dynamical systems whose states represent compactly supported measures in which progression in time corresponds to progressively increasing magnification. Application of the ergodic theorem will show that, generically, dimension conservation is valid. This 'almost everywhere' result implies a non-probabilistic statement for homogeneous fractals."

Coming back to his work on random walks and Lie groups in 1971 he published 'Random walks and discrete subgroups of Lie groups" [15] in which he gave a new application of probability theory to group theory. A discrete subgroup Γ of a non-compact connected Lie group G is called a *lattice* if the quotient space G/Γ has finite measure. This means that, there is a subset $D\subset G$ with finite left-invariant Haar measure and the translates $D\gamma, \gamma\in\Gamma$ cover G. Hillel's main result was that a lattice subgroup of $\mathrm{SL}(d,\mathbb{R})$ cannot be isomorphic to a subgroup of $\mathrm{SL}(2,\mathbb{R})$ (discrete or not). His proof goes via the study of the maximal boundaries and the Poisson boundaries. He constructs a random walk on a lattice subgroup Γ of $\mathrm{SL}(d,\mathbb{R})$ whose Poisson boundary coincides with the maximal boundary of $\mathrm{SL}(d,\mathbb{R})$. This work is related to the well-known Mostow rigidity and the methods of Hillel influenced the work of G. Margulis on super-rigidity.

At the retirement conference for G.A. Hedlund, one of the pioneers of the modern study of topological dynamics, Hillel presented his proof of the unique ergodicity of the horocycle flow. This was published in the following year [16] in the proceedings of that meeting. To explain this result we need to define what is the horocycle flow and what is unique ergodicity. A real flow (X,T_t), here the acting group is the reals, is said to be *uniquely ergodic* if it has a unique invariant measure. This measure is ergodic and the system has a unique minimal set. The restriction of the action to the minimal set is strictly ergodic. As a consequence it follows that all orbits of the flow are equidistributed.

The horocycle flow has for its state space the unit tangent bundle of a 2-dimensional surface with constant negative curvature. The universal cover of such a space is the hyperbolic plane. In the unit disc model for the hyperbolic plane the geodesics are described by circular arcs that are perpendicular to the boundary. The horocycles are circles tangent to the boundary. Algebraically one can describe this as follows. Let $G=\mathrm{SL}(2,\mathbb{R})$ and Γ a cocompact discrete subgroup. Let $X=G/\Gamma$ and T_t denote the one parameter subgroup $\begin{pmatrix}1&t\\0&1\end{pmatrix}$ which acts on X by multiplication on the left. This is one of the two horocycle actions, the other is described by $\begin{pmatrix}1&0\\t&1\end{pmatrix}$. Hedlund had shown that these horocycle flows were minimal and Hillel proved that they are also uniquely ergodic — hence strictly ergodic. This result was the first in a long series of results classifying the invariant measures for actions by unipotent groups on homogeneous spaces. The major work here was done by Marina Ratner and the theory that she developed, called simply Ratner theory has been widely employed in applications of homogeneous dynamics to number theory and geometry.

In 1975 the Israeli Institute for Advanced Studies was established at the Hebrew University of Jerusalem. This institute has no permanent members, but serves as the venue for semester-long, or year-long, programs of intensive international activity in a specialized area. In the first year of the Institute, 1975–1976, Hillel was among the organizers of the program in ergodic theory. This included Donald Ornstein, from the US, Jean-Paul Thouvenot from France, Mike Keane from Holland, and others. There were also many visitors and among them was Konrad Jacobs, who was the leading figure in ergodic theory in Germany in the preceding period. At that time he was more interested in combinatorics and he gave a colloquium talk in which he presented E. Szemerédi's proof of the Erdős–Turan conjecture on arithmetic progressions in sets of integers with positive density.

Having learned about the result Hillel went on to give a completely different proof using methods from ergodic theory [17]. The first part of the new proof is formulating a result in ergodic theory which is equivalent to Szemerédi's theorem. This result is a far reaching generalization of the Poincaré recurrence theorem, it reads:

Theorem 2. *If $(X,\mathcal{B},\mu,T)$ is a measure preserving system and $A\in\mathcal{B}$ has positive measure then for any positive integer k there is some $n>0$ such that*

$$\mu(A\cap T^n(A)\cap T^{2n}(A)\cap\cdots\cap T^{kn}(A))>0.$$

Poincaré's theorem is the case $k=1$ and this is called the multiple recurrence theorem. The fact that one can deduce Szemerédi's theorem form this multiple recurrence theorem was immediately clear to Hillel because it follows from what is now called the 'Furstenberg correspondence principle' which Hillel first exploited to great benefit in his thesis on prediction theory. This principle gives a precise rendering of the heuristics whereby the integers are seen as a measure space, density of a set representing its measure and the shift operation, $x \mapsto x+1$, a measure preserving transformation. The existence of an arithmetic progression $\{a+id,\ i=0,1,\ldots,k\}$ in a set of positive density takes on the meaning of a point in a set of positive measure recurring in the set after a power of the measure preserving transformation is applied k times in succession.

In the other direction it is straightforward to see that the combinatorial theorem implies the ergodic version. Having established this equivalence the big task remained of proving this multiple recurrence theorem. The first reduction is to use the ergodic decomposition of an arbitrary measure preserving transformation to an integral of ergodic systems. It remains then to establish the theorem for ergodic systems.

On the one hand it is quite easy to establish it for the Kronecker systems, which correspond to rotations on a compact group. At the other extreme are the *weakly mixing* systems. These are systems for which the unitary operator associated to the measure preserving transformation has no non trivial eigenfunctions. An equivalent definition is that the direct product of the system with itself is ergodic under the diagonal action. For these weakly mixing systems Hillel proved a multiple ergodic theorem, generalizing von Neumann's mean ergodic theorem. The multiple recurrence then follows from this.

The really difficult task was combining these two extreme kinds of behavior. For this Hillel developed a structure theorem for all ergodic systems. The first part is a measure theoretic analogue of distality. A *measure distal* system is one which can be represented as a tower of isometric extensions beginning with a Kronecker system. This is modeled on his structure theorem for distal transformations in the setting of topological dynamics. Next he generalized the notion of weakly mixing from simple systems to their extensions. A measure preserving system $(X,\mathcal{B},\mu,T)$ is an *extension* of a measure preserving system $(Y,\mathcal{C},\nu,S)$ if there is a measurable mapping $\pi : X \to Y$ mapping μ to ν and satisfying $\pi T = S\pi$. The extension is *relatively weakly mixing* if the relative product of X with itself over Y is ergodic. The exact definition of relative product is somewhat technical and I will omit it. Hillel's structure theorem can then be formulated as follows (A similar result was obtained independently at the same time by R. Zimmer).

Theorem 3. *An arbitrary ergodic system is a relatively weakly extension of a measure distal system.*

Using this Hillel was able to fully prove the multiple recurrence theorem. Shortly afterwards the new methods showed their strength in the proof of a new combinatorial result on subsets with positive density in $\mathbb{Z}^d$. This work which was done jointly with Yitzhak Katznelson [24] was based on a generalization of the multiple recurrence theorem in which the powers of the transformation T were replaced by a general collection of $k+1$ commuting measure preserving transformations $\{T_0,T_1,\ldots,T_k\}$.

1980s – today

Hillel gave a marvelous exposition of these and many other results in his monograph *Recurrence in Ergodic Theory and Combinatorial Number Theory* [18], which was based on the Porter Lectures that he gave at Rice University in 1978. This monograph opened up an entire new field in the interface between dynamical systems and combinatorial number theory. Over the next ten years in a series of papers written jointly with Katznelson the new methods were able to prove many new combinatorial theorems. The paper which culminated the series titled 'A density version of the Hales–Jewett theorem' [26] established a density version of the combinatorial Hales–Jewett theorem, much as Szemerédi's theorem is a density version of the classic van der Waerden theorem.

Here is what the theorem says. One generalizes the notion of an arithmetic progression to something purely combinatorial. Let $WN(A)$ be the set of all words of length N over the alphabet $A=\{a_1,a_2,\ldots,a_k\}$. For any $x \in A$ consider a word $w \in WN(A)$ in which x occurs at least once. Let $L=\{w(a_1),w(a_2),\ldots,w(a_k)\}$ where $w(a_j)$ is obtained from $w(x)$ by replacing all copies of x by a_j. Then L is called a *combinatorial line*.

Theorem 4. *For any $\epsilon>0$ and integer k there exists an $M(\epsilon,k)$ such that if $N>M(\epsilon,k)$ any set $S \subset WN(A)$ with cardinality $|S|>\epsilon N^k$ contains a combinatorial line.*

Here is a more geometric version which can be easily deduced from the above.

Theorem 5. *There is a function $Q(\epsilon,q)$ defined for $\epsilon>0$ and q a prime power, so that if F is the field with q elements and V is a vector space over F of dimension $N>Q(\epsilon,q)$, and if $S \subset V$ is a subset with cardinality $|S|>\epsilon q^N$, then S contains an affine line.*

This latter theorem was established several years earlier in [25]. These results have been vastly extended over the years by many people, including Vitaly Bergelson [2] who was a student of Hillel. Bergelson's achievements include replacing arithmetic progressions by patterns based on polynomials of higher degree. In another development inspired by Hillel's structure theorem for ergodic systems a more refined structure theorem was found which involves special types of isometric extensions involving compact nilmanifolds. This was done in joint work of Bernard Host and Bryna Kra [28], and independently by Tamar Ziegler [34], who was also a student of Hillel. The role that the nil-systems play in the study of what have become to be known as non-conventional ergodic averages was originally discovered by Hillel and me, and appeared for the first time in the papers by J.-P. Conze and E. Lesigne [5,6]. It was elaborated and applied to polynomial patterns in sets of positive density in [27].

Hillel returned to homogeneous dynamics in his paper titled 'Stiffness of group actions' [19]. In it he introduced the new notion of *stiffness*. If ν is a probability measure on a group G then an action of G on a space X is *ν-stiff* if every ν-stationary measure on X is invariant. Hillel showed that for carefully chosen ν on $\mathrm{SL}(d,\mathbb{Z})$, namely probability measures ν so that the corresponding stationary measure on the boundary of $\mathrm{SL}(d,\mathbb{R})$ is absolutely continuous with respect to Lebesgue, the action of $\mathrm{SL}(d,\mathbb{Z})$ on $\mathbf{T}^d$ is ν-stiff. He conjectured that this should be true for any measure whose support generates $\mathrm{SL}(d,\mathbb{Z})$. This insight was more than confirmed about ten years later in [3,4], and further greatly generalized in [1].

In [22] Hillel and Glasner embark on a general study of stationary dynamical systems, that is actions of a group

G with a probability measure m and an action of G on a space X equipped with an m-stationary measure ν. They develop a general theory of factors, extensions and conditional measures. They prove a structure theorem and use it to establish a theorem of Szemerédi type for the group $\mathrm{SL}(2,\mathbb{R})$. In a further paper [23] they use the structure theorem to prove a version of multiple recurrence for sets of positive measure in a general stationary dynamical system.

Hillel came back to his ideas about fractals and ergodic theory in a wonderful series of lectures at Kent State University which appeared in [21]. In it he develops a theory of *mini-sets* and *micro-sets* of a closed subset A of $\mathbb{R}^d$. A mini-set of A is just the intersection of A with a small square that is re-scaled to be of unit size, while the micro-sets of A are the limits in the Hausdorff metric of mini-sets of A. There is a new notion of dimension called the star-dimension and ergodic theory is used to show that the star-dimension of a set A is the maximal Hausdorff dimension of a micro-set of A. There are further results connected to preservation of dimension for homogeneous fractals and connections with ergodic theory.

I have tried to give a survey of some of the host of original ideas that are to be found in the publications of Hillel Furstenberg. This published work resembles the tip of an iceberg in the following sense. Generations of students and colleagues have benefited from his ideas spanning a range of mathematics going far beyond the areas I have touched on in the above. In addition his work has inspired countless developments which I have barely touched on. In conclusion I would like to thank Hillel for all that I have learned from him during all these years from 1963 until today.

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