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History

A look at my paper with Hans, almost forty years later

It has been ten years since Hans Duistermaat passed away. To commemorate this sad and sudden loss for the Dutch mathematics community, *Indagationes Mathematicae* dedicates the first issue of 2021 to the mathematical legacy of this extraordinary mathematician. At *Nieuw Archief voor de Wiskunde* we asked his collaborator and dear friend Alberto Grünbaum to look back on their joint paper 'Differential equations in the spectral parameter'. With the help of some personal letters from Hans' own hand, he tells us how this landmark paper came into being and discusses some problems and research directions that originated from it.

The paper with Hans

The year 1981 was a very special one for me. Hans Duistermaat came to teach a class in Berkeley and at the end of November I gave a seminar on some problem in analysis motivated by signal processing. Hans attended the talk and a few days later told me that he had found a tool that might help us solve the problem I had posed. We spent one full afternoon working with the new tool at a coffee house and failed to find any solutions besides the classical ones that I had mentioned in my talk: they dealt with Bessel and Airy functions.

Hans left for Holland shortly after that and in a few days sent me a letter, dated 15 December 1981, 'proving' my conjecture that there was nothing beyond these classical cases. Fortunately my conjecture was wrong and even Hans could make small computational mistakes as we will see.

In the meantime I had been doing these computations not by hand, as Hans had, but using the symbol manipulator Maxima (the only one that I still use). I had obtained a certain nonlinear ode that had to be solved and after some more playing around and looking for its rational solutions I almost fell to the floor when I saw that the first nontrivial rational solution of the Korteweg-de Vries equation satisfied our nonlinear ode. I then tried the next few ones in the hierarchy contained in a by then recent paper [1]. And I could not believe that we were so lucky. I wrote to Hans immediately, just before the end of the year and eventually got an answer by mail dated 27 January 1982.

After this a long period followed that included several visits of mine to Odijk, where I would stay with Hans and his wonderful family at their home. I would show to Hans

Dear Alberto,

I think that I have proved your conjecture, that there is no other equation than Bessel which allows for a differential equation in λ of order ≤ 4. I followed your idea of expanding in powers of $x - \gamma_i$, γ_i = the roots of the denominators in $\alpha(x)$. In principle I get $c_7 \cdot (x-\gamma_i)^{-7} + c_6 \cdot (x-\gamma_i)^{-6} + c_5 \cdot (x-\gamma_i)^{-5} + \ldots$, but c_7 and c_5 are automatically zero, and the conclusion that Bessel is the only case can be drawn by considering only $\underbrace{c_4 \text{ and } c_3}_{\text{involve only the quadratic terms in the } \alpha'}$ (c_6 and c_4 are not sufficient).

I hope you can check the calculations (and maybe you can write a similar program to do the case $m = 5$ on MACSYMA ...). However, to be honest, I don't find these calculations very inspiring. They only increase my feeling that we are missing some (geometric or computational ...) insight.

I hope you had an interesting trip to France!

With cordial greetings,

Hans.

Datum Jan 27, 1982
Uw kenmerk
Ons kenmerk
Onderwerp

Dear Alberto,

Thank you very much for your inspiring letters of Dec 31 and Jan 5. I received them both a week ago, the mail is really a shame. The connections between our problem and KdV contain still a lot of mystery for me.

Dear Alberto,

When I ~~was~~ suggested that you write the article under your name only, then it is not because I consider it of too low level, but because you have worked so much more on it than me! If I would think harder and read more, I could maybe reach a better understanding (However, I have been so lazy with this project!)

where our project was going and he would listen with interest and make insightful remarks but kept saying that he was tied up with other things. When it was time to write things up he tried to push me to do it all by myself, and I protested saying that this was a joint effort from the beginning. His remarks are in a letter from September 1983.

In response to this letter I made a simple proposal: you come to Berkeley for one month and stay at my home; if you get excited again then we write a joint paper, otherwise I go my own way.

Hans came to Berkeley with his tennis racket and swimming shorts but he did not get to use them even once. We would start working at 8 am and finish almost with no interruptions at 10 pm, day in and day out for an entire month. I had succeeded in getting him fully back into the project, and his newly found enthusiasm was a thing to watch. In a few more months and by regular mail we had what I thought was a decent paper. I was ready to publish and I am sure that Hans would have preferred to keep working till "we had nothing else to say". But I prevailed. I wrote an introduction explaining what had motivated me to undertake this problem. Hans objected to this introduction in a letter dated 30 November 1984.

30 November 1984

Dear Alberto,

Yesterday I got your letter from Paris, concerning the Introduction, p. 2 from the middle, top of p. 3, p. 5 l. 13–16.

I did not comment too much on it when you were in Odijk, but I am still unhappy with the extensive reference on p. 2 and the top of p. 3 to the "miraculous existence of a second order differential operator which commutes with a certain integral operator". You now propose even more of that: "here again the possible existence of a local operator commuting with the integral one is of paramount importance".

My objection is that we do <u>not</u> consider the problem of finding commuting operators in this paper. More or less as a consequence, our results cannot (yet) be applied to that problem. Furthermore, I think it is not wise to emphasize too much that we do not apply our theory to the problems which motivated us. I once got a paper rejected ~~partially~~ among others on the ground that I did not apply the results to the problem which motivated originally the subject of the paper.

My preference is to mention that the question arose in your studies of ~~the~~ "limited angle tomography", followed by a reference. Then mention that in ~~such~~ (or related?) problems one has to study eigenfunctions $\varphi(x,\lambda)$ of more general second order differential operators as a function both of x and λ, on some sort of "equal footing". Here again one or two references.

Because we do not have the aforementioned applications, I think it is better to, basically, <u>let our question and its answer stand on its own</u>.

Things that came later

The main point of this note is to show that although he was correct in objecting, after all these years my initial motivation has yielded some fruits. It is clear to me that without Hans this entire enterprise would not have succeeded. He was extremely inspirational, and even now after so many years when I am looking for answers to some of my questions I think of asking him first.

On the one hand the 'bispectral problem' (as the work that we did with Hans is now known) has been considered in many different incarnations, and by several different people. They are too may to mention here. Let me just say that there are not only continuous-continuous versions (as Hans and I had) but also discrete-continuous as well as discrete-discrete ones. There are scalar as well as matrix valued versions.

On the other hand there has been significant progress in the original problem of 'time-and-band limiting', i.e. in the construction of a differential operator of low degree that commutes with a naturally appearing integral one. Many years ago I managed to push this in the case of the 'even family' in the terminology of [6], see [7]. But the KdV case resisted my efforts for a long time. This has recently changed completely as one can see in [3, 4]. For matrix valued versions of work that can be traced back to my paper with Hans, see for instance [2, 8–11].

An example

I will include now a single example. This is in the spirit of our paper with Hans, whose last section is a collection of examples which I have found extremely useful over many years. There are many other examples produced in the same fashion.

It shows how the KP hierarchy, see [12], which contains the KdV one a special case gives examples of integral operators for which one can explicitly construct commuting differential operators. This was the ultimate goal in this project that I undertook with Hans a long time ago. The fact that its taken such a long time to reach this goal, makes Hans' reluctance to talk about these things back in 1986 well justified.

I consider an instance of the KP hierarchy resulting from the $\tau(x)$ function (they were called $\theta(x)$ in our paper) given by

$$\frac{x^4}{12}+\frac{x^3}{3}+x^2-p_3x+2x-p_3+p_2^2+1.$$

This gives rise to a bispectral situation: the wave function constructed out of τ satisfies a differential equation in x (of order four) with eigenvalue given by z^4. The wave function is built in the manner discovered by M. Sato [14] back in 1981, the year of Hans' visit to Berkeley. This topic was later considered by people like G. Segal and G. Wilson [15] and many others. The τ function given above is an appropriate linear combination of characters, along the lines of M. Sato. In all of this my debt to the work of G. Wilson, see [17,18] is huge.

The coefficients of the differential operator are rational functions of x and they are too messy to record here.

It also satisfies a differential equation in this spectral parameter of the form indicated in the paper with Hans and coefficients $b_i(z)$ given by

$$b_0 = \frac{(p_3-2)z^3+(2-2p_2)z^2-2}{z^5},$$
$$b_1 = -\frac{(p_3-p_2{}^2-1)z^3+2z-2}{z^3},$$
$$b_2 = -\frac{(p_3-2)z^3+2z-2}{2z^3},$$
$$b_3 = \frac{z^2-1}{3z^2},$$
$$b_4 = \frac{1}{12},$$
$$b_5 = \frac{1}{60}.$$

Now we come to the really new result: out of this τ function one builds a kernel for an integral operator that acts on $L^2(t,\infty)$ and depends on a parameter s.

The kernel of this integral operator is built in a canonical way from the τ function at hand.

This integral operator can be seen to commute with a differential operator of the form

$$R_{s,t}(z,\partial_z) = \sum_{m=0}^{4} \partial_z^m f_m(z)\,\partial_z^m,$$

with coefficients given as follows

$$\begin{aligned}
f_0(z) &= s^2z^2(s^6z^6-4ts^6z^5+6t^2s^6z^4-64s^4z^4\\
&\quad -4t^3s^6z^3+176ts^4z^3+t^4s^6z^2-168t^2s^4z^2\\
&\quad +456s^2z^2+64t^3s^4z-672ts^2z-8t^4s^4\\
&\quad +240t^2s^2-96),\\
f_1(z) &= -4s^2(z-t)(s^4z^7-3ts^4z^6+3t^2s^4z^5\\
&\quad -36s^2z^5-t^3s^4z^4+60ts^2z^4-24t^2s^2z^3\\
&\quad +84z^3-12tz^2-12t^2z-12t^3),\\
f_2(z) &= 6(z-t)^2(s^4z^6-2ts^4z^5+t^2s^4z^4-16s^2z^4\\
&\quad +8ts^2z^3+4t^2s^2z^2+4z^2+8tz+12t^2),\\
f_3(z) &= -4z^2(z-t)^3(s^2z^3-ts^2z^2-4z-4t),\\
f_4(z) &= z^4(z-t)^4.
\end{aligned}$$

The reader will notice that the differential operator does not depend on the free parameters p_2, p_3.

A question that is still open

In our paper there are two families of solutions to our problem. The first one involves the characters of representations of $\mathrm{GL}(N)$. For the second family we identify appropriate τ functions. Is there a group floating around?

My huge debt to the Dutch

I came into the problem of 'time-and-band-limiting', see [13,16] because of my work on X-ray tomography. Here one tries to reconstruct a function in $\mathbb{R}^2$ from its line integrals. Few people know that this problem (for $\mathbb{R}^3$) was apparently first considered by Hendrik Lorentz, the celebrated Dutch physicist. See page 3 of [5] for more references. I learned of this from one of my teachers at Rockefeller University, George Uhlenbeck, another distinguished Dutch physicist who in 1925 — the same year of his paper with S. Goudsmit on the spin of the electron — extended the work of Lorentz to an arbitrary dimension. At Rockefeller I wrote a thesis, as a student of H. P. McKean, on some mathematical problems arising from the Boltzmann equation of statistical mechanics. If I remember correctly Hans wrote a PhD thesis on statistical mechanics too. Boltzmann had as one of his students Paul Ehrenfest who went to Leiden because Lorentz was there, and Uhlenbeck was a student of Ehrenfest from whom he inherited his life long interest in statistical mechanics. Please notice that in this short description of my debt to the Dutch I have not mentioned Korteweg and his student de Vries.

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