Ross J. Kang
Department of Mathematics
Radboud University Nijmegen
ross.kang@gmail.com

The Solution

Parties a smidgen smaller

Recently Ashwin Sah, a precocious researcher at MIT, made small but significant progress in an old, difficult and important problem in combinatorics. Ross Kang describes this problem and the steps that have been made thus far.

There's been a rush of exciting results in Combinatorics recently, with many evidently immutable challenges suddenly yielding. When I was asked to write about one of these, I elected for what on the face of it appears to be relatively small progress, but for one of the oldest, most cherished, and most difficult challenges of them all.

Imagine hosting a party prior to the 9th of March of this year. Of course, as a conscientious host, you naturally wonder to yourself, are there 3 guests here all of whom have previously shaken hands with each other, or are there 3 no two of whom have ever shaken hands? Could it be that this is already guaranteed merely by dint of having sufficiently many houseguests? More specifically, what is the *minimum* number (if it exists) of houseguests to guarantee this, arguably curious, property?

The answer here is not 5, as you could have some 5 houseguests sat around a round table, each having shaken hands only with his or her two neighbours. And a routine case analysis will check that the minimum is indeed exactly 6, so perhaps a subdued affair. On the other hand, already when the parameter 3 above is raised to, say, 5, the precise value of the corresponding minimum is a well-known open problem — the current best lower bound is 43, while the upper bound was recently lowered with extensive computer assistance to 48 by Angeltveit and McKay [1]. You quickly run out of room in your house, because the problem suffers so badly from 'combinatorial explosion'.

The fact that when 3 is replaced by any fixed positive integer k, the corresponding minimum, call it $R(k)$, is always well-defined was shown by a Cambridge polymath of the early twentieth century, Frank Plumpton Ramsey (or 'The man who thought too fast' [6], according to a recent *New Yorker* piece). He established this fact as a lemma: he needed it for the decidability of some first-order logic statements in relation to a Hilbert problem [7]. This has become one of his namesakes: the $R(k)$ are known as *Ramsey numbers*.

One might better cast this in graph theoretic terms: associate a vertex of the graph to each guest; if two guests have ever shaken hands, include an edge between their respective vertices. Ramsey proved that, given a positive integer k, there is some least integer $R(k) < \infty$ such that in *any* graph on $R(k)$ vertices, one is guaranteed some k-vertex subset within which either all $\binom{k}{2}$ pairs of vertices are included as edges — it forms a *clique* — or none are — it forms a *stable set*. See Figure 1 for an example showing $R(4) > 17$. Cliques and stable sets are two of the tidiest graph structures possible, so one can view Ramsey's theorem as finding order within some arbitrary system, a common refrain in mathematics.

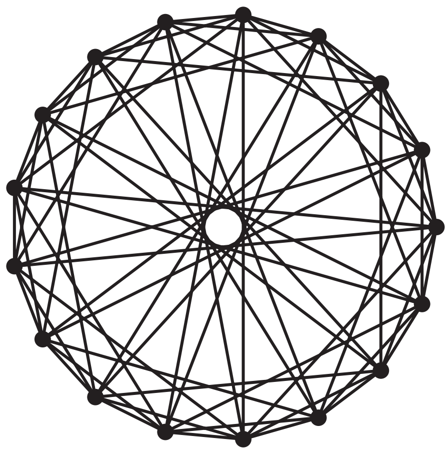


Figure 1 A Payley graph of order 17 has no cliques or stable sets of size 4.

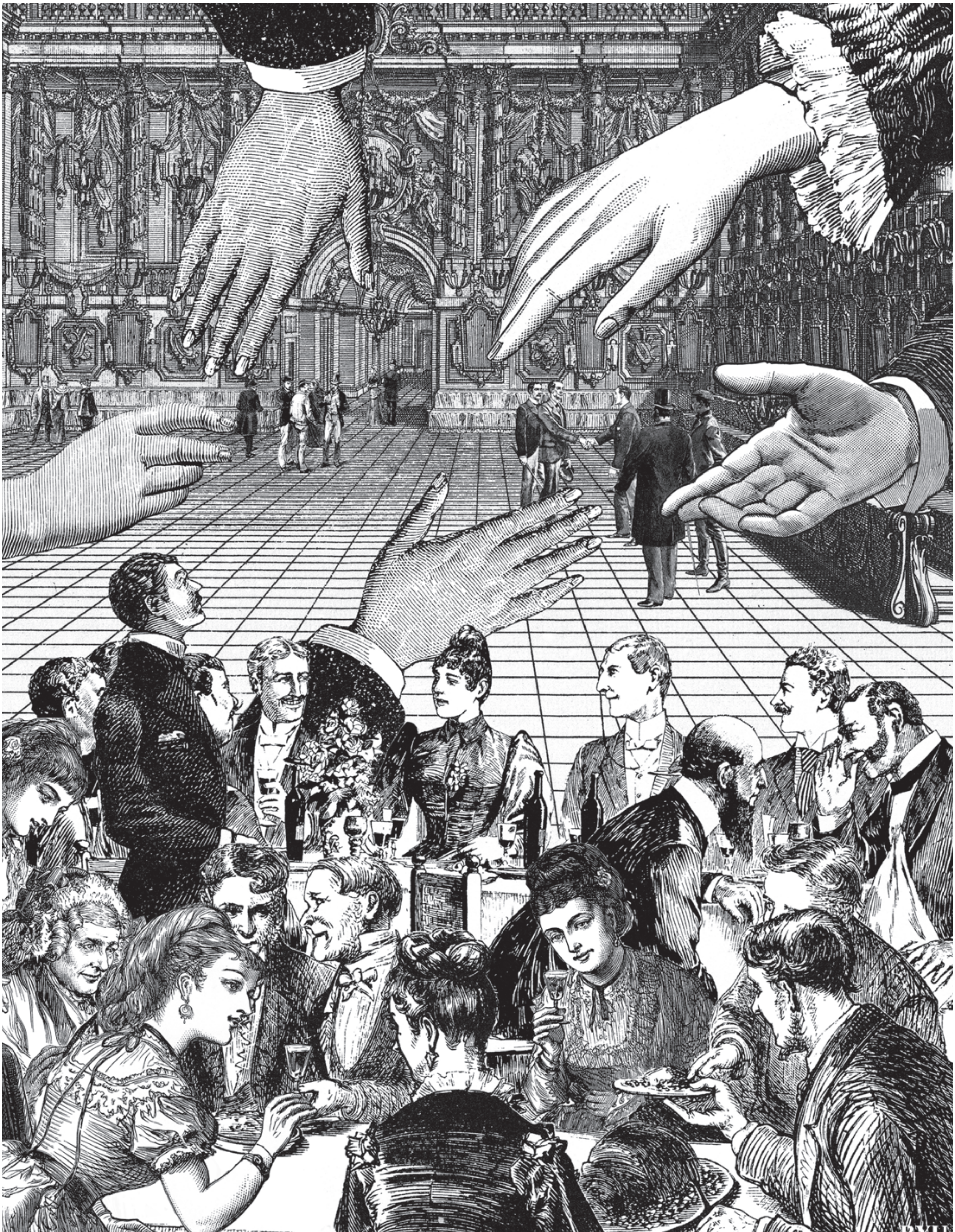


Illustration: Ryu Tajiri

While Ramsey was satisfied with the fact that each $R(k)$ exists (nevertheless in a footnote he wrote "But this value is, I think, still much too high."), Paul Erdős and George Szekeres [5] soon after revisited Ramsey's theorem, as one way towards a problem in discrete geometry, the fabled *Happy Ending Problem*. (Their work appeared in one of the first issues of *Compositio Mathematica*.) They produced the following argument for bounding $R(k)$. First, consider a more general definition, where $R(k,l)$ denotes the least integer such that in any graph on $R(k,l)$ vertices, one is guaranteed a clique of order k or a stable set of order l. Second, note that it is enough to verify, together with the trivial base cases $R(1,k)=R(k,1)=1$ and $R(2,k)=R(k,2)=k$, that $R(k+1,l+1)\le R(k,l+1)+R(k+1,l)$ for each $k,l\ge 1$. For then this class of upper bounds satisfies the recurrence for the binomial coefficients $\binom{k+l}{k}$. Third, take any graph with $R(k,l+1)+R(k+1,l)$ vertices, so that any vertex v must have either $R(k,l+1)$ neighbours or $R(k+1,l)$ non-neighbours, by the pigeonhole principle. In the former case, there is by definition a set of neighbours of v that forms either a k-vertex clique — so together with v is a $(k+1)$-vertex clique — or an $(l+1)$-vertex clique, and either way we are done. The latter case about $R(k+1,l)$ non-neighbours is symmetric. This yields that

$$R(k)=R(k,k)\le\binom{2k-2}{k-1}=\left(\frac{1}{2\sqrt{\pi}}+o(1)\right)\frac{4^k}{\sqrt{k}} \qquad (1)$$

as $k\to\infty$. Upon first witnessing such a short and basic argument, one might ask: why can't this easily be beaten? But before that, the bound being exponential in k, could the Ramsey numbers really be so large?

A little over a decade later, Erdős addressed this latter question [3] and in doing so incidentally seeded the birth of two separate, though closely intertwined fields, namely Probabilistic Combinatorics and Random Graph Theory. In hindsight, his unexpected realisation was natural, that is, to use a 'most arbitrary' of systems, a binomial random graph. That is, take a collection of n vertices and, independently, for each of the $\binom{n}{2}$ pairs, toss a fair coin to determine whether or not the pair is included as an edge. By a straightforward estimation of the size of a largest clique or stable set in this probability space of graphs, together with a flourish of what is now called the *probabilistic method*, he proved the existence of graphs certifying an exponential lower bound:

$$R(k)\ge\left(\frac{1}{\sqrt{2}\,e}+o(1)\right)k\sqrt{2}^k$$

as $k\to\infty$. After more than seventy years and the growth of a mature, consistently active field around this result, which incorporates other perspectives from nearby areas such as probability, algorithms and network science, this remains, up to the leading constant, the best known lower bound.

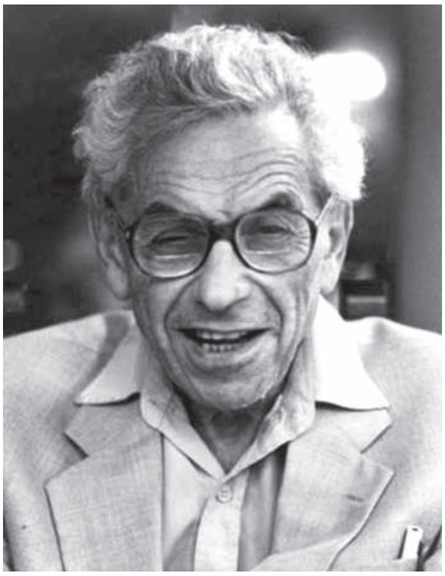
Back to the question of improving upon the simple bound (1), it also turns out to be a stubbornly enduring challenge. However, a natural idea for enhancing the Erdős–Szekeres inductive argument above was introduced by Andrew Thomason, also of Cambridge, in the late eighties [9]. Instead of considering some vertex v, one may consider some r-vertex clique (or, symmetrically, r-vertex stable set) and bound the number of ways it can be extended to a $(k+1)$-vertex clique (respectively $(l+1)$-vertex stable set) inductively. The catch is that to conclude, instead of using the pigeonhole principle as before, one must consider graphs that contain neither a $(k+1)$-vertex clique nor an $(l+1)$-vertex stable set, so-called *Ramsey graphs*, and carefully estimate how many r-vertex cliques and stable sets they contain. Thomason leveraged the case $r=3$ to obtain a polynomial factor improvement on (1).

A good while later, we saw the last major advance. I recall it was my first professional visit to Hungary in 2006 when Ron Graham — the late mathemagician and juggler extraordinaire — had completely rejigged his plenary talk to tell the story of Ramsey numbers and, in particular, to eagerly discuss a circulating preprint by a then-Cambridge-PhD-student David Conlon. It eventually appeared in the *Annals of Mathematics* [2]. Conlon, building comprehensively upon the enhanced inductive strategy of Thomason's, achieved a first superpolynomial improvement upon (1):

$$R(k)\le\binom{2k-2}{k-1}\exp(-c(\log k)^2/\log\log k)$$

for some absolute constant $c>0$. Conlon

Frank Ramsey



Paul Erdős

Photo: Cheryl Graham

Ron Graham

Ashwin Sah

did so by carrying out the clique and stable set counts for larger fixed r via a *quasirandom* framework: roughly speaking, by comparing Ramsey graphs to binomial random graphs.

Now fourteen years since the last breakthrough, it is another phenom, this time a young scholar based in *another* Cambridge (Massachusetts), having only this year completed his bachelor's degree, yet with already an astonishing array of scientific contributions under his belt, Ashwin Sah. By incorporating techniques from the theory of Graph Limits — a modern, analytic descendant of Random Graph Theory — Sah has accomplished a more efficient, and nearly optimal, estimate of the required subgraph counts, effectively pushing the quasirandom framework to its limit [8]. This gives a further superpolynomial improvement on (1):

$$R(k) \le \binom{2k-2}{k-1} \exp(-c(\log k)^2)$$

for some absolute constant $c > 0$.

So where are we now? At broader granularity, the current bounds yield

$$\sqrt{2} \le \liminf_{k\to\infty} R(k)^{1/k} \le \limsup_{k\to\infty} R(k)^{1/k} \le 4,$$

but this is how it stood in 1947! Erdős [4] wrote, "I offer 250 dollars for a proof that

$$\lim_{k\to\infty} R(k)^{1/k} = c \tag{2}$$

exists and I offer 10 000 dollars for a disproof. I am of course sure that (2) holds. I offer 250 dollars for the determination of c... perhaps $c = 2$?" Sah's progress has shown that there is some give, albeit tiny in relative terms, in this famous challenge. Will it precipitate sudden further progress? Or will it amount to an even more difficult barrier?

This article is dedicated in admiration of the life and achievements of Ron Graham (1935–2020).

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