In de verdediging

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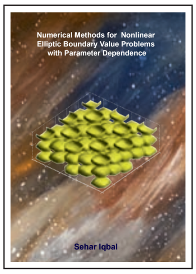

Numerical Methods for Nonlinear Elliptic Boundary Value Problems with Parameter Dependence

Sehar Iqbal

In January 2020 Sehar Iqbal from Utrecht University successfully defended her PhD thesis with the title *Numerical Methods for Nonlinear Elliptic Boundary Value Problems with Parameter Dependence*. Sehar carried out her research under the supervision of Prof. dr. R.H. Bisseling and dr. P.A. Zegeling.

During her PhD Sehar worked on numerical methods for nonlinear elliptic partial differential equations with parameter dependence, such as the Gelfand–Bratu model, and singularly perturbed convection-diffusion-reaction equations. After completing her PhD Sehar got interested in and is searching for a job in Data Science.

Nonlinear elliptic partial differential equations

Second order PDEs have the general form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + F = 0.$$

Suppose first that all the coefficients are constant, in this case such a PDE is called elliptic if

$$B^2 - AC \leq 0.$$

Examples of elliptic PDEs are the Poisson equation $\Delta u = u_{xx} + u_{yy} = f(x,y)$, for some function f, with the Laplace equation $\Delta u = u_{xx} + u_{yy} = 0$ as a special case. In general, the coefficients of the PDEs don't have to be constant. In case they are linear functions, the PDE is called elliptic if the principal symbol is non-zero away from the origin. In case they are non-linear functions, then the PDE is called elliptic if it is elliptic at every solution u; where being elliptic at a solution u means that it's linearization is elliptic at u.

Nonstandard finite difference schemes

The majority of differential equations (both linear and nonlinear) can't be solved exactly in terms of elementary functions. As a consequence, a variety of numerical methods has been constructed to find approximate solutions. One of the traditional numerical methods is the finite difference method (FD). Unfortunately, numerical instabilities may arise when using finite difference approximations.

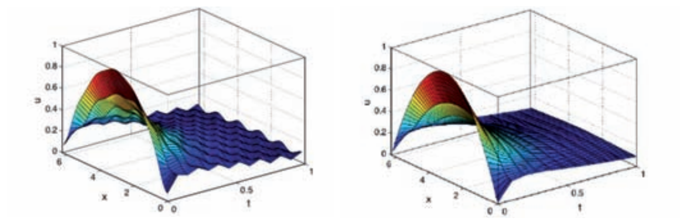

Figure 1 Numerical solutions using a standard FD method and a nonstandard FD method. The solution on the right remains stable (nonstandard), whereas the solution on the left possesses numerical instabilities (standard).

Pas gepromoveerden brengen hun werk onder de aandacht. Heeft u tips voor deze rubriek of bent u zelf pas gepromoveerd? Laat het weten aan onze redacteur.

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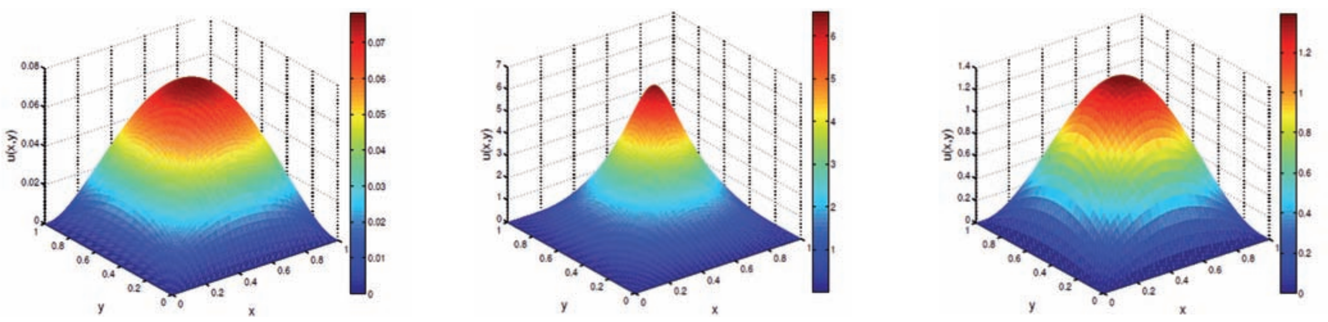

Figure 2

Normally, numerical instabilities can be recognized as the solutions to the discrete finite difference equations that do not correspond to any solution of the original differential equations. To eliminate or minimize such difficulties so-called nonstandard finite difference methods (NSFD) have been developed.

The Gelfand–Bratu model

The Gelfand–Bratu model is an elliptic partial differential equation. Traditionally, the names Bratu and, sometimes, Gelfand are coupled to this model. However, in historical literature, Picard was the first one who actually introduced this model in 1898, showing it does not possess non-unique solutions. The Gelfand–Bratu model simulates a thermal reaction process in a rigid material, where the process depends on a balance between chemically generated heat addition and heat transfer by conduction. Originally, the Gelfand–Bratu problem appears in the study of the quasilinear parabolic problem

$$T_t = \Delta T + \lambda(1-\epsilon T)e^{\frac{T}{1+\epsilon T}}, \quad x \in \Omega \subset \mathbb{R}^n, n = 1,2,3,$$
$$T = 0, \quad x \in \partial\Omega.$$

Note that for $\lambda = 0$ we obtain the heat equation. Here $\lambda > 0$ is known as the Frank–Kamenetskii parameter, T is the temperature, Δ is the Laplacian, $\frac{1}{\epsilon}$ is the activation energy, t is time, m is the mass of a single molecule and Ω is a bounded domain (for example, a bounded container).

PDEs of this kind appear also in other applications and models, such as fluid dynamics at high Reynolds numbers, aerodynamics, atmospheric models, in elasticity theory (membrane buckling), in astronomy (gravitational equilibrium of polytropic stars and the Chandrasekhar model of the expansion of the universe), in thermo-electro-hydrodynamic models and in nanotechnology (electrospinning processes).

In her research Sehar focused on numerical methods for this type of PDEs. For the Gelfand–Bratu model numerical results indicate the existence of infinitely many solutions. These solutions are either periodic or semi-periodic. In her thesis Sehar demonstrates that the convergence of all solutions, namely, unique, lower, upper, periodic and semi-periodic, is obtained for small values of the dependence parameter λ. Particularly, the non-linear multigrid full approximation storage (FAS-MG) method is found to be more efficient than the other nonlinear multigrid methods.

Furthermore, the numerical bifurcation behavior of the Gelfand–Bratu problem in three dimensions shows the existence of two new turning points. For even higher dimensions, numerical experiments show the existence of several types of solutions. Bifurcation curves confirm the theoretical results of the higher-dimensional Gelfand–Bratu problem as presented in the literature. Moreover, in her thesis Sehar proposes a higher order non-uniform finite difference grid in order to solve singularly perturbed boundary value problems with steep boundary-layers. She also sheds light on various theoretical properties concerning the extremum values and the asymptotic value at the right boundary point. In this line of research accuracy of the numerical schemes is essential. In her research 6th-order accuracy is established by considering only three-point central non-uniform finite differences. Numerical results illustrate that to achieve the 6th-order of accuracy, the proposed method needs approximately a factor of 5–10 fewer grid points than the uniform case.

In Figure 2 we present two solutions of the Gelfand–Bratu model in $\mathbb{R}^2$ for $\lambda = 1$ (left and middle panel) and the unique solution close to the critical value $\lambda = \lambda_c \approx 6.808124423$ (right panel) using the non-linear multigrid full approximation storage (FAS-MG) method on 161×161 grid points.

The more personal aspect

Behind all dissertations there is always a person, with flesh and bones, who has endured the long path of a PhD trajectory and has produced the work at hand.

Were you also involved in some other activities and events during your PhD?

"I really enjoyed participating in the meetings and conferences of the Dutch–Flemish Scientific Computing Society (SCS). The SCS has as a goal to unite researchers in and users of numerical mathematics and scientific computing. With some PhD colleagues at the Mathematical Institute at Utrecht University we also started the SIAM Student Chapter, an initiative which aims at increasing interaction among PhD and master students working in applied mathematics. This was in 2017. I served as secretary of this SIAM Student Chapter for two years (2017–2019).

Morreover, I am an active member of European Women in Mathematics-Netherlands (EWM-NL), a platform for all female professional mathematicians working in the Netherlands. EWM-NL forms the Dutch branch of EWM."

Would you like to share some memories from the last four years?

"During the last four years I interacted with many people from different parts of the world, I learned a lot of this interaction. It was an amazing experience to present my research work in local and international conferences. The most fascinating part of my PhD was the teaching I had to do for the mathematics programs at Utrecht University. I taught and assisted in many mathematics courses."