Problemen

| Problem Section

This Problem Section is open to everyone; everybody is encouraged to send in solutions and propose problems. Group contributions are welcome. We will select the most elegant solutions for publication. For this, solutions should be received before **15 April 2020**. The solutions of the problems in this issue will appear in the next issue.

Problem A (proposed by Hendrik Lenstra)

For every positive integer n, we write $[n] := \{0, 1, \ldots, n-1\}$. For every integer m, we let $T(m) := m(m+1)/2$ be the m-th triangular number. Let $\tau : [n] \to [n]$ be the map given by $m \mapsto T(m) \bmod n$.

a. For which n is τ a permutation?

b. For these n, determine the sign of T as a function of n.

Problem B (proposed by Onno Berrevoets)

Let G be a finite group of order n. A map $f : G \to \mathbb{R}$ is called a near-homomorphism if for all $x, y \in G$, we have $|f(xy) - f(x) - f(y)| \leq 1$.

a. Show that for every near-homomorphism f from $G \to \mathbb{R}$, we have $\operatorname{diam}(f[G]) := \sup_{x,y \in G} |f(x) - f(y)| \leq 2 - 2/n$.

b. Show that if G is cyclic, then there exists a near-homomorphism $f : G \to \mathbb{R}$ with $\operatorname{diam}(f[G]) = 2 - 2/n$.

Problem C (proposed by Hendrik Lenstra)

Let $n \geq 4$ be an integer and let A be an abelian group of order 2^n. Let σ be an automorphism of A such that the order of σ is a power of 2. Then the order of σ is at most 2^{n-2}.

Edition 2019-4 Last time, we forgot to credit Thijmen Krebs for his solution to problem B. We apologize for the oversight. This time, we received solutions from Alex Heinis, Alexander van Hoorn, Thijmen Krebs and Hendrik Reuvers.

Problem 2019-4/A (proposed by Onno Berrevoets)

Let $F \in \mathbb{Z}[X]$ be a monic polynomial of degree 4. Let $A \subset \mathbb{Z}$ with $\#A \geq 5$ be such that for all $a \in A$ we have $2^{2019} \mid F(a)$. Prove that there exist distinct $a, a' \in A$ such that $a \equiv a' \bmod 2^{202}$.

Solution We received solutions from Alexander van Hoorn, Thijmen Krebs and Hendrik Reuvers. Both Thijmen and Hendrik realized that the statement is still true when we replace 202 by 505; this solution is based on their arguments.

For all $a, a' \in A$ with $a \neq a'$, we find $F(a') - F(a)$ is divisible by $a' - a$. Fix $a_1 \in A$ and let $F_1(x) = \frac{F(x) - F(a_1)}{x - a_1}$. If $a - a_1$ is divisible by 2^{505} for some $a \in A_1 := A \setminus \{a_1\}$, we are done. Otherwise, $F_1(a)$ must be divisible by $2^{2019-504} = 2^{1515}$ for all $a \in A_1$. This means that $F_1(x)$ is a monic polynomial of degree 3 for which there exists a set A_1 of cardinality at least 4 such that for all $a \in A_1$ we have $2^{1515} \mid F_1(a)$.

We can repeat this construction, each time reducing the size of A by one and reducing the exponent of 2 by 504. In the end, we either find that there exist $a, a' \in A$ such that $a - a'$ is divisible by 2^{505}, or there exists a monic constant polynomial F_4 together with a set A_4 of cardinality at least 1 such that for all $a \in A_4$, we have $F_4(a)$ is divisible by $2^{2019 - 4 \cdot 504} = 2^3$. Since $F_4 = 1$ is not divisible by 8, clearly the latter cannot be the case. So there exist $a, a' \in A$ such that $a - a'$ is divisible by 2^{505}.

For the sake of completeness, our original solution was based on taking the determinant of the matrix $(a^i)_{a \in A, i \in \{0,1,\ldots,4\}}$ and noting that first of all, this is divisible by the product

Redactie: Onno Berrevoets, Rob Eggermont en Daan van Gent

problems@nieuwarchief.nl

www.nieuwarchief.nl/problems

Oplossingen

| Solutions

of all the $a-a'$ with $a,a' \in A$, and secondly, we can change the fourth row into $(F(a))_{a\in A}$ by row operations that do not change the determinant. Since this row is divisible by 2^{2019}, so is the product of the $a-a'$. A similar method works if the polynomial F has at least one odd coefficient, even if F is not monic.

Problem 2019-4/B (proposed by Jan Draisma)

Let $n \in \mathbb{Z}_{\geq 2}$ and let S be a non-empty subset of $\{1,2,\ldots,n-1\}$.

1. Prove that there exists an n-th root of unity $z \in \mathbb{C}$ such that the real part of $\sum_{i\in S} z^i$ is smaller than or equal to $-\frac{1}{2}$.
2. Suppose that n is not a power of 2. Prove that the $-\frac{1}{2}$ in the previous part is optimal in the following sense: there exists a non-empty subset S of $\{1,2,\ldots,n-1\}$ such that for all n-th roots of unity z, the real part of $\sum_{i\in S} z^i$ is at least $-\frac{1}{2}$.

Solution The first part is proved as follows. Let A be the circulant $n\times n$ matrix with $A_{i,j}=1$ if $i-j\in S$ (taken mod n) and $A_{i,j}=0$ otherwise. Observe that A has an orthonormal basis of eigenvectors $\mathbf{v_z}=\frac{1}{\sqrt{n}}(1,z,z^2,\ldots,z^{n-1})$ (with respect to the standard inner product on $\mathbb{C}^n$), where the z run over the n-th roots of unity. This uses the fact that $\sum_{i=1}^{n} z^i = 0$ whenever z is an n-th root of unity not equal to 1 and n otherwise. The eigenvalue of $\mathbf{v_z}$ is precisely the expression $\lambda_z=\sum_{i\in S} z^i$.
Note that if $\mathbf{x}=\sum_z c_z \mathbf{v_z}$, then

$$\mathrm{Re}(\mathbf{x}^* A\mathbf{x}) = \mathrm{Re}\left(\sum_z \lambda_z c_z^* c_z\right) \geq \min_z \mathrm{Re}(\lambda_z)\,\mathbf{x}^*\mathbf{x}.$$

Let $i\in S$ and consider $\mathbf{x}=\mathbf{e}_1-\mathbf{e}_{i+1}$. We easily verify that

$$-1 \geq \mathbf{x}^* A\mathbf{x} \geq \min_z \mathrm{Re}(\lambda_z)\cdot 2.$$

So there exists an n-th root of unity $z\in\mathbb{C}$ such that the real part of $\sum_{i\in S} z^i$ is smaller than or equal to $-\frac{1}{2}$.
The solution to the second part is based on the solution by Hendrik Reuvers. If $n=pm$ with p an odd prime, we take $S=\{m,2m,\ldots,\frac{p-1}{2}m\}$. Let z be an n-th root of unity. If $z^m=1$, clearly we are done. Otherwise, note that $z^{km}=\overline{z^{(p-k)m}}$. We find the real part of $2\sum_{s\in S} z^s$ equals $\sum_{i=1}^{p-1}(z^m)^i=-1$, since we are taking the sum of all primitive p-th roots of unity. This gives the desired inequality (or equality in this case).

Problem 2019-4/C (proposed by Onno Berrevoets)

Let $n\geq 1$ be an integer. Denote by S_n the permutation group on $\{1,\ldots,n\}$. An element $\sigma\in S_n$ is called *representable* if there exists a polynomial $f\in\mathbb{Z}[X]$ such that for all $x\in\{1,\ldots,n\}$ we have $\sigma(x)=f(x)$. What is the number of representable elements of S_n?

Solution We received solutions from Alex Heinis, Thijmen Krebs and Hendrik Reuvers. This solution is based on the solution by Alex.
Suppose $\sigma\in S_n$ is representable, say by the polynomial $f\in\mathbb{Z}[X]$. Observe that $f(n)-f(1)$ is divisible by $n-1$. It follows $\{f(1),f(n)\}=\{1,n\}$. Suppose $f(1)=1$. Observe that $f(n-1)-f(1)$ is divisible by $n-2$, and hence we can conclude $f(n-1)=n-1$. Repeating this, we find $f(x)=x$ for all $x\in\{1,2,\ldots,n\}$. If $f(1)=n$, we replace it by $n+1-f$ and apply the same procedure to find $f(x)=n+1-x$ for all $x\in\{1,2,\ldots,n\}$. So we find that when $n\geq 2$, there are exactly two representable permutations, namely the identity and the permutation that sends x to $n+1-x$. When $n=1$, these two coincide, so there is only one representable permutation in this case.