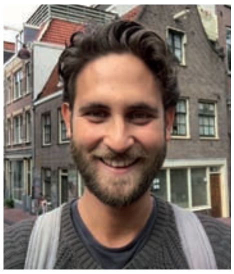
In de verdediging

| In defence



Explicit Serre Weights for Two-Dimensional Galois Representations
Misja Steinmetz

In August 2019 Misja Steinmetz from King's College London successfully defended his PhD thesis with the title *Explicit Serre Weights for Two-Dimensional Galois Representations*. Misja carried out his research under the supervision of prof. dr. F.I. Diamond. Fred Diamond did his PhD with Andrew Wiles in Princeton in 1988, which makes Misja Andrew Wiles' mathematical grandson! Since the first of September Misja is working at the Mathematical Institute in Leiden in a postdoc position.

During his PhD Misja worked on generalisations of Serre's modularity conjecture to totally real number fields, for example $\mathbb{Q}(\sqrt{2})$. This research can be thought of as a part of the (mod p) Langlands programme, a famous international research programme, which has recently gotten a lot of attention due to the fact that two Fields Medals and the Abel Prize in 2018 were awarded to researchers in this area.

The conjecture

In 1987 Fields medalist and Abel Prize winner Jean-Piere Serre posed an important conjecture (now a theorem) relating mod p Galois representations and modular forms. This line of research in number theory dates to 1973 and it all started with a conjecture made by Serre in a letter to John Tate. The conjecture that Serre made in his letter to Tate is quite remarkable, so remarkable that Tate replied one month later enthusiastically to Serre's letter. We read in his reply to Serre's letter:

> "Please give me a little more time [...] on the elliptic curve report. I have started it, but have been distracted by various things, in particular, your conjectures on $\rho : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F}_{\ell^a})$."
>
> John Tate, 2 July 1973

Essentially, Serre's conjecture says that any mod p Galois representation satisfying some straightforward properties (it is irreducible and the determinant of complex conjugation is equal to -1) must be the 'shadow' of a modular form. A modular form is a (complex) analytic function on the upper half-plane satisfying a certain kind of functional equation with respect to the group action of the modular group, and also satisfying a growth condition. Mod p Galois representations played a crucial role in Andrew Wiles' proof of Fermat's last theorem.

Wiles' proof of Fermat's last theorem consists of establishing a formal link between modular forms and elliptic curves, however only readers who have studied the proof more closely will know that mod p Galois representations play an important role. These are continuous group actions of the absolute Galois group of $\mathbb{Q}$ on a vector space over $\mathbb{Z}/p\mathbb{Z}$; you can think of them as algebraic

Pas gepromoveerden brengen hun werk onder de aandacht. Heeft u tips voor deze rubriek of bent u zelf pas gepromoveerd? Laat het weten aan onze redacteur.

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shadows containing enough information to compare the two objects (in this case modular forms and elliptic curves) by comparing their shadows. In fact, a proof of Serre's conjecture would also have sufficed for Wiles to prove Fermat, but possibly this seemed even more out of reach to him than the proof that he eventually gave.

In Serre's letter to Tate we also read that Serre was already daydreaming about generalisations to more general global fields and, even, to higher dimensional representations.

The generalisation

In his research Misja looked into generalisations of Serre's conjecture to totally real number fields, for example $\mathbb{Q}(\sqrt{2})$. Instead of considering actions of the absolute Galois group of $\mathbb{Q}$ on a vector space over $\mathbb{Z}/p\mathbb{Z}$, he considers actions of the absolute Galois group of $\mathbb{Q}(\sqrt{2})$, and instead of modular forms, he considers Hilbert modular forms. A Hilbert modular form is a generalisation of a modular form to functions of two or more variables. It is a (complex) analytic function on the m-fold product of upper half-planes satisfying a certain kind of functional equation.

Suppose you consider a mod p Galois representation satisfying Serre's properties. It may be claimed this representation appears as the 'shadow' of any modular form whatsoever. This would lead to a correct, but very imprecise conjecture because there are infinitely many modular forms that would have to be checked before potentially falsifying this conjecture. Already in 1987 Serre put a lot of work into using the characteristics of the Galois representation to obtain a finite list of possible modular forms giving this representation, which gave rise to the so-called strong version of Serre's conjecture. In 1992 our very own Bas Edixhoven showed that the strong version implies the weak version of the conjecture.

Analogously, in generalisations of the conjecture to totally real fields like $\mathbb{Q}(\sqrt{2})$ the real challenge is to come up with a good way to reduce to a finite list of modular forms. Until 2016 the only way in which such a finite list could be produced was by making use of the abstract and non-explicit theory of p-adic Hodge theory. In his thesis, building on previous work of Dembele–Diamond–Roberts and Calegari–Emerton–Gee–Mavrides, Misja gives an alternative way of producing this finite list making use of explicit local class field theory only. One way to state the main theorem of his thesis, therefore, would be to say: "There exists an explicit reformulation of the strong version of Serre's conjecture for totally real fields, which is equivalent to the conjecture formulated in terms of p-adic Hodge theory." Misja's results, for example, make it much easier to do computer calculations on these objects.

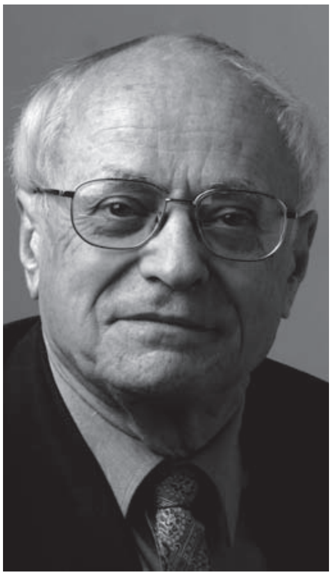
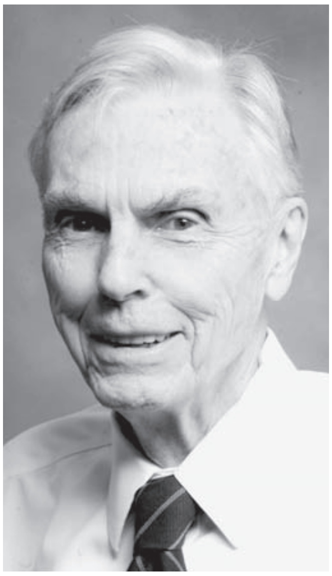


Photo: Anne Lise Flavik

Jean-Pierre Serre



John Tate

Applications of generalisations of Serre's conjecture

Analogously to applications of Serre's conjecture to Diophantine equations like the Fermat equation, it is natural to wonder whether generalisations of the conjecture have similar applications. Indeed, the so called *modular approach* to Diophantine problems turns out to be rather powerful, and generalisations of Serre's conjecture can help with establishing a link to modular forms. However, since for totally real fields other unconditional methods are readily available such as the modularity of many elliptic curves and level lowering theorems, it is usually not necessary to make results conditional on an unproved generalisation of Serre. For general number fields, when much less is known, conditioning on generalisations of Serre is often the best available option. A recent theorem for example states (conditional on a version of Serre's conjecture) that the Fermat equation with prime exponent does not have solutions in certain imaginary quadratic fields. This gives a beautiful illustration of applications of these conjectures to other areas of number theory.

The more personal aspect

Behind all dissertations there is always a person, with flesh and bones, who has endured the long path of a PhD trajectory and has produced the work at hand.

Misja, are there any stories you would like to share from your time as a PhD student?

"People often exaggerate by saying a book has changed their life, but in my case I think I am justified in saying this. When I was 16 I was unsure whether to study mathematics or physics. I asked my mathematics teacher who said there were very exciting times ahead in physics with the Large Hadron Collider, and he didn't know about similarly interesting things being done in mathematics. Exactly around this time I read Simon Singh's book called *Fermat's Last Theorem*. The story of Wiles' discovery blew me away! I realised how exciting mathematical research could be, and chose to study mathematics. A combination of pure coincidence and a fascination that never faded resulted in my submitting a PhD thesis a few months ago, roughly 13 years after reading the book, on a topic that is not so far removed from Wiles' original breakthrough.

Another funny incident occurred just before the ICM in Rio, an unknown mathematician sat down next to me. After a while it turned out he had been to the previous ICM in Seoul already, so I asked him what I should expect. What I understood from his story was that they give you a medal, which I imagined to be an award for participating, just like in a marathon. Later someone told me that I had been talking all night to Artur Avila who won a Fields Medal in 2014 in Seoul."