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Research

A formula for the permanent

In January 2003, Jaap Spies found a solution to 'Problem 29' of the old NAW Problem Section. The solution was the permanent of a square matrix. Moments later, he discovered an algorithm for the calculation of the permanent with basic algebraic means.

Permanent? Yes, permanents are a long time with us. And as the name suggests they will stay. In 1812 permanents were introduced independently by Binet and Cauchy writing about 'certain functions permanente'. See Minc [5] for a complete history up to 1978. Most people know the permanent from a conjecture (now theorem) of Van der Waerden (1926) about double stochastic matrices, but permanents have a broad range of applications: counting problems in combinatorics, in graph theory the number of perfect matches can be calculated by means of the permanent of an incidence matrix, applications in statistics, even in chemistry and many many more. See [3,5].

The permanent of an $m \times n$ matrix A with $m \leq n$ is defined by

$$\operatorname{per}(A) = \sum_{\pi} a_{1\pi(1)} a_{2\pi(2)} \cdots a_{m\pi(m)}, \qquad (1)$$

where the summation extends over all one-to-one functions π from $\{1,\ldots,m\}$ to $\{1,\ldots,n\}$. In case A is a square matrix, we get the well-known formula for a permanent, which is like the determinant but without the signs as the sum of all $n!$ diagonal products.

A hard problem

Binet gave algorithms for $m = 2,3$ and 4. There are general algorithms by Binet/Minc and Ryser. See [3,5]. For square matrices there are a few published algorithms other than Ryser's, based on the finding of a certain coefficient of a certain term in a certain polynomial. K. Balasubramanian [1] uses Muir-algebra in his 1980-thesis. Bax and Franklin [2] use 'finite differences' to come to a formula. David Glynn [4] finds the formula [4, Theorem 2.1]:

$$\operatorname{per}(A) = \frac{\left[\sum_{\delta} \left(\prod_{k=1}^{n} \delta_k\right) \prod_{j=1}^{n} \sum_{i=1}^{n} \delta_i a_{ij}\right]}{2^{n-1}},$$

where the outer sum is over all 2^{n-1} vectors $\delta = (\delta_1 = 1, \delta_2, \ldots, \delta_n) \in \{\pm 1\}^n$.

An elementary approach

We define a vector $\bar{x} = (x_1, x_2, \ldots, x_n)^T$ and a vector $\bar{y}$ with $\bar{y} = A\bar{x}$. Let P be a multivariate polynomial defined by

$$P(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} y_i = \prod_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_j. \qquad (2)$$

All terms of P have degree n. Expanding this polynomial and summing the terms with $x_1 x_2 \ldots x_n$ we get a well known result

$$\begin{aligned} &\sum_{\pi} a_{1\pi(1)} x_{\pi(1)} \cdot a_{2\pi(2)} x_{\pi(2)} \cdot \cdots \cdot a_{n\pi(n)} x_{\pi(n)} \\ &= \left(\sum_{\pi} a_{1\pi(1)} a_{2\pi(2)} \cdots a_{n\pi(n)}\right) \cdot x_1 x_2 \cdots x_n \\ &= \operatorname{per}(A) \cdot x_1 x_2 \cdots x_n. \end{aligned}$$

A theorem

First we define a polynomial Q of degree $2n$:

$$Q(x_1, x_2, \ldots, x_n) = \left(\prod_{i=1}^{n} x_i\right) \cdot P(x_1, x_2, \ldots, x_n). \qquad (3)$$

Theorem. *The permanent of A is*

$$\operatorname{per}(A) = \frac{\sum_{x_1 = 1,\, x_j = \pm 1,\, j > 1} \prod_{i=1}^{n} x_i \prod_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_j}{2^{n-1}}. \qquad (4)$$

Proof. We sum $Q(x_1, x_2, \ldots, x_n)$ over all possible $x_i = \pm 1$.

$$\begin{aligned} &\sum_{x_i = \pm 1} Q(x_1, x_2, \ldots, x_n) \\ &\quad = \sum_{x_i = \pm 1} (x_1 x_2 \ldots x_n) P(x_1, x_2, \ldots, x_n). \end{aligned}$$

As we can easily see only the term $\operatorname{per}(A) \cdot x_1 x_2 \cdots x_n$ is always contributing to this sum because $x_i^2 = 1$ for $i = 1, 2, \ldots, n$. Terms of $Q(x_1, x_2, \ldots, x_n)$ with factor x_k missing in $P(x_1, x_2, \ldots, x_n)$, are counted once t and once $-t$ so the overall result is 0. There are 2^n possible vectors $\bar{x}$ with $x_i = \pm 1$, so we have proved:

$$\operatorname{per}(A) = 2^{-n} \cdot \sum_{x_i = \pm 1} Q(x_1, x_2, \ldots, x_n).$$

By symmetry we can symplify this result by fixing $x_1 = 1$

$$\operatorname{per}(A) = 2^{-n+1} \cdot \sum_{x_1 = 1, x_j = \pm 1, j > 1} Q(x_1, x_2, \ldots, x_n). \qquad \square$$

It is easy to see that formula (4) is essentially the same as Glynn's. A preliminary form of formula (4) was first published in 2006 as a sidenote in [7]. The formula fits in the series from Balasubramanian [1], Bax-Franklin [2] to Glynn [4] as described in the Wikipedia page 'Computing the permanent' [8].

References

1 K. Balasubramanian, *Combinatorics and Diagonals of Matrices*, PhD dissertation, Indian Statistical Institute, Calcutta, 1980.

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3 R.A. Brualdi, H.J. Ryser, *Combinatorial Matrix Theory*, Cambridge University Press, 1991.

4 D. Glynn, The permanent of a square matrix, *European Journal of Combinatorics* 31(7) (2010), 1887–1891.

5 H. Minc, *Permanents*, Addison-Wesley, 1978.

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7 J. Spies, The Dancing School Problems. A permanent solution of Problem 29, *NAW* 5/7(4) (2006), 283–285.

8 Computing the permanent, en.wikipedia.org/wiki/Computing_the_permanent, 2019.