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Mathematical modeling with measures

In December 2018 a workshop entitled 'Mathematical Modeling with Measures: where Applications, Probability and Determinism Meet' was held at the Lorentz Center in Leiden. This workshop brought together mathematicians from countries in Eastern and Western Europe and North and South America with different expertise including, modeling, analysis and probability to discuss state of the art research results for a common mathematical language that is used in these fields, which is modeling with measures, and to initiate new collaborations in an attempt to solve some open problems that were posed by the workshop participants. Azmy Ackleh, Rinaldo Colombo, Paola Goatin, Sander Hille en Adrian Muntean report about this event.

Having in mind the use of measures as key modeling tool, workshop participants considered a wide variety of applications: from structured population models, to selection-mutation models, to vehicular/traffic traffic flows, to balance equations, to probability, ...

A selection model

Let us begin with a simple model studied two decades ago by Ackleh et al. in [1]. Therein, the authors consider the integro-differential equation on the state space of continuous functions:

$$\frac{d}{dt}x(t,q) = x(t,q)\,[q_1 - q_2X(t)]. \qquad (1)$$

Here, $x(t,q)$ is the density of individuals having trait $q=(q_1,q_2)\in Q=[a_1,b_1]\times[a_2,b_2]$ a rectangle in the interior of $\mathbb{R}^2_+$, where q_1 describes the growth rate and q_2 describes the mortality rate. This model implies that individuals with trait q produce individuals with the same trait (i.e., pure selection) and that individuals compete for resources since the mortality term is dependent on the level of the total population $X(t)=\int_Q x(t,q)\,dq$. By letting q^* be the unique trait value attaining $\max_{q\in Q} q_1/q_2 = b_1/a_2$ and using the Lyapunov function $L(t)=x(t,q)^{1/q_1}/x(t,q^*)^{1/q_1^*}$ one can show that $\frac{d}{dt}L(t) = (q_2^*/q_1^* - q_2/q_1)\,X(t)\,L(t)$. Since, $(q_2^*/q_1^* - q_2/q_1) < 0$, establishing boundedness of $X(t)$, one can deduce that for any ε radius ball in Q centered at q^* denoted by B_ε, $\int_{Q/B_\varepsilon} x(t,q)\,dq \to 0$, as $t\to\infty$. Using this result one can show that $x(t,q)\to \frac{b_1}{a_2}\delta_{q^*}$ in the weak* topology. Biologically this implies that the fittest trait is q^* and the long term dynamics of this model is that of competitive exclusion where the fittest trait survives and all other traits go to extinction. However, from a mathematical point of view it is clear that this Dirac limit $\frac{b_1}{a_2}\delta_{q^*}$ is not an element of the state space considered in [1] which is $C(Q)$ but is an element of the more natural space for formulating this model which is the space of finite signed measure $M(Q)$. Indeed, the authors in [2] reformulated a more general model which includes (1) as a special case on the space of finite signed measures.

Shepherd dogs and fairy tales

A recurrent intriguing question is: how can a leader drive a multitude towards a given goal?

With respect to the multitude, the leader can be *attractive* or *repulsive*, for instance. It is easy to refer to these two sample cases respectively as to the *pied piper* that attracts mice (e.g. *De rattenvanger van Hamelen*, see [9]) or to a shepherd dog herding sheep. On this basis, a variety of new control problems can be formalized. Can we characterize the *best* strategy for the leader? Clearly, here *best* may mean fastest, or cheapest, or simplest, or ...

Further questions arise when we start thinking at a team of leaders cooperating towards the same goal. How much are 2 shepherd dogs more efficient than only one? Does there exist an *optimal* number of pied pipers to gather mice in a given region?

The next step is even more intriguing. Indeed, we may pass from a control problem, where one goal has to be achieved, to a *game*, where different goals are sought by different (teams of) competing leaders. Does there exist a *winning* strategy? Can we characterize Nash [12] equilibria? (These are strategies that each controller adopts because other choices would be less convenient, see also [6].)

When we get to the formalization of these problems, we find an exciting wealth of tools that mathematics offers to describe these situations. During the meeting at the Lorentz center, focus was mostly on descriptions based on conservation laws. Call $\rho = \rho(t,x)$ the time (t) and space (x) dependent density of individuals, think at ρ as at the number of mice/sheep per square meter. Let $P_1(t),\dots,P_n(t)$ be the time dependent positions of the pipers/shepherd dogs and describe the pipers-mice or dogs-sheep interactions through the speed $v = v(t,x,\rho,P_1,\dots,P_n)$. We thus describe the whole dynamics through the continuity equation

$$\partial_t \rho + \operatorname{div}(\rho v(t,x,\rho,P_1,\dots,P_n)) = 0$$

while the controls or strategies are the speeds $u_1,\dots,u_n$ of the leaders, so that

$$P_i(t) = P_i^o + \int_0^t u_i(\tau)\,d\tau \quad \text{with} \|u_i\| \le U,$$

U being the maximal leaders' speed and P_i^o the initial position.

Basic well posedness and stability results for the resulting problem were obtained in [7]. The goals of the leaders can be easily formalized through suitable integrals of ρ.

At the time of this writing, the existence and the characterization of optimal controls is an open problem, as also any information on Nash equilibria. We expect that these questions have to be answered prior to suggesting optimal escape strategies to mice and sheep...

Nash and Braess close roads

The Braess Paradox, see [4] is a famous example, showing that the dynamics on networks can significantly differ from that

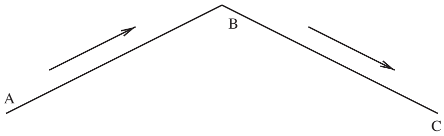

Figure 1

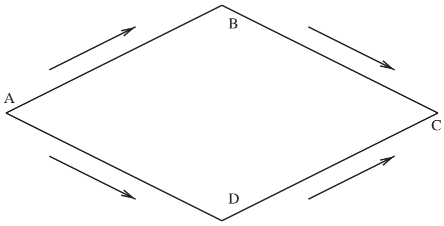

Figure 2

along a single road, displaying surprising properties.

Cities A and C are connected by two roads that pass through city B (here and in what follows all roads are one way). The travel time along AB depends on traffic, say it equals the number of vehicles traveling along that road divided by 100. On the contrary, the road from B to C is so wide that the travel time along this segment is 45, regardless of traffic. See Figure 1. Clearly, if 4000 commuters need to go from A to C every day, their travel time is $\frac{4000}{100} + 45 = 85$.

New roads are build, so that A is connected to C also through D. To ease computations, we assume that the road AD has the same travel time as that between B and C, while the road DC is identical to AB. See Figure 2. How will our 4000 commuters distribute between the two routes? Clearly, the evident symmetry between ABC and ADC suggests that half will go through B and half through D. The resulting travel times are

$$\begin{array}{ccc} \mathrm{AB} + \mathrm{BC} = \mathrm{ABC} & & \mathrm{AD} + \mathrm{DC} = \mathrm{ADC} \\ \frac{2000}{100} + 45 = 65 & \text{and} & 45 + \frac{2000}{100} = 65. \end{array}$$

Note that this configuration is an example of a Nash equilibrium. Indeed, look at each driver as to a *player*, whose objective is to minimize his/her travel time. Imagine that one of the commuters passes from using route ABC to route ADC. Then, the new travel times become

$$\begin{array}{ccc} \mathrm{AB} + \mathrm{BC} = \mathrm{ABC} & & \mathrm{AD} + \mathrm{DC} = \mathrm{ADC} \\ \frac{1999}{100} + 45 < 65 & \text{and} & 45 + \frac{2001}{100} > 65, \end{array}$$

showing that this change of mind is not convenient for our commuter.

Due to the intensive use of this network, a new road is now built connecting B to D. This brand new road allows for infinite speed, so that commuters reach D from B in time 0. See Figure 3. Now, to go from A to C three choices are available: ABC, ABDC and ADC. How will commuters distribute among these three different routes? In other words, is there a new Nash equilibrium? And which is it?

A moments' thought reveals that if x commuters travel along ABC, y along ABDC and z along ADC, then the three travel times are

$$\begin{aligned} \mathrm{ABC} &= \mathrm{AB}+\mathrm{BC} &&= \frac{x+y}{100} + 45 \\ \mathrm{ABDC} &= \mathrm{AB}+\mathrm{BD}+\mathrm{DC} &&= \frac{x+y}{100} + 0 + \frac{y+z}{100} \\ \mathrm{ADC} &= \mathrm{AD}+\mathrm{DC} &&= 45 + \frac{y+z}{100}. \end{aligned}$$

The equality of the travel times yields $x+y = y+z = 4500$, which is inconsistent with the total number of commuters being $x+y+z = 4000$. Thus, no equilibrium exists when all routes are used. (Here, an *equilibrium* configuration is a distribution of drivers yielding the same travel time along all used routes, see for instance [6].)

A reasonable guess is now that the Nash equilibrium prior to the construction of BD still is a Nash equilibrium in presence of BD. But it is not the case because it is convenient for a driver to pass from, say, route ABC to route ABDC. Indeed, the travel time corresponding to $x = 2000$, $y = 0$ and $z = 2000$ is 65 along both routes ABC and ADC. On the other side, setting $x = 1999$, $y = 1$ and $z = 2000$ results in the travel times

$$\mathrm{ABC} = 65,\ \mathrm{ABDC} = 40.01,\ \mathrm{ADC} = 65.01.$$

Thus, we imagine that as soon as BD is opened to commuters, they find convenient to use it. As more and more commuters drive along the new route ABDC, the corresponding travel time increases, but remains lower than those along ABC and ADC. Thus, the new Nash equilibrium corresponds to all commuters driving along ABDC, that is $x = 0$, $y = 4000$ and $z = 0$.

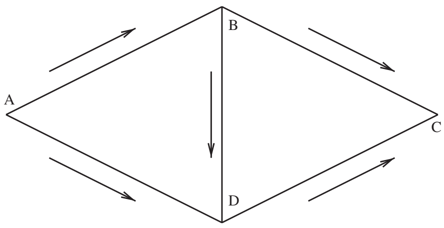

Figure 3

Here comes the paradox: the new travel time is $\frac{4000}{100}+0+\frac{4000}{100}=80$! It is *higher* than prior to the construction of BD!

In other words, adding a road to a network may make the network less efficient, even if the new road is, by itself, extremely efficient (our BD segment is traveled at infinite speed!).

This remark is clearly counter-intuitive, paradoxical, but it is *real*. Here, we refer to the famous closing of 22nd street in New York that took place on 22 April 1990, see [11], and to [3]. The reader is invited to personally search for other real examples.

The literature on Braess paradox [4] is vast and currently comprises a variety of phenomena not necessarily related to traffic on road networks. During the meeting at the Lorentz center, some discussions centered about the possibility that PDE based macroscopic models are able to capture this paradox. A first result in this connection is [6], but several questions remain unanswered. For instance, can the dynamics of PDE describe the *insurgence* of a Braess-like regime in a network? It goes without saying that this descriptive ability is preliminary to tackling optimal management problems.

A somewhat *inverse* Braess paradox is reported to happen in flocks of sheep passing through narrow gates [8], see also https://www.youtube.com/watch?v=mkqhYhdgvGg. Similarly, Hughes [10] suggested that an obstacle, suitably placed in front of an exit, may increase the through flow of pedestrians, thus reducing evacuation time. Indeed, even if the presence of obstacles may be seen as leading to a worse condition, it may reduce the inter-pedestrian pressure near the exit and prevents it from blocking. This effect opens interesting perspectives for safety and efficient evacuation of buildings and other confined environments. It is thus of great importance to develop mathematical models capable to describe these phenomena. In this perspective, various studies [13] showed that models based on the classical mass conservation equation

$$\partial_t \rho + \mathrm{div}(\rho\, V(\rho)\, \vec{\mu}) = 0, \qquad (2)$$

where $\vec{\mu}$ is the vector field of the trajectories followed by pedestrians (possibly dependent on the density distribution through an eikonal equation), are not able to capture this behavior. One needs either to add a momentum balance equation describing acceleration, or to account for inter-pedestrian interactions through non-local terms. In the latter case, the velocity vector field is given in the form

$$\vec{\mu}(t,x) = \vec{v}(x) - \varepsilon \frac{\nabla(\rho * \eta)}{\sqrt{1+\|\rho * \eta\|^2}}, \qquad (3)$$

where $\vec{v}$ is the vector field of the preferred path, for example the shortest to destination, tempered by the latter non-local term, which pushes pedestrians towards low density regions through the action of a suitable positive mollifier η (here $\rho * \eta$ is the usual convolution product), accounting for the local density distribution.

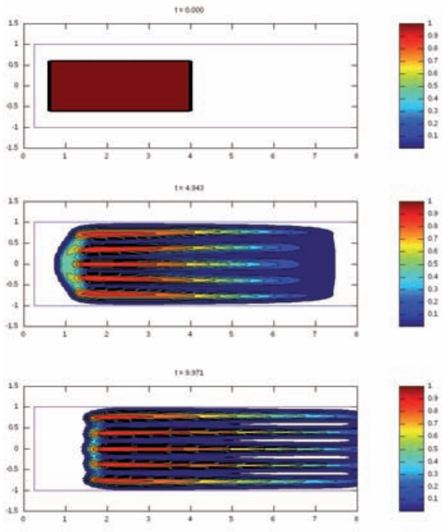

Figure 4

Model (2), (3) is able to display not only Braess paradox but also lane formation [5]. Indeed, Figure 4 shows the movement of a crowd. Initially, individuals are uniformly distributed in the rectangle above. At time $t=0$, they move to the right, the ones in front being faster. Without any ad hoc prescription, individuals form five lanes of higher density (see [5] for details about the numerical integration).

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