Problemen

| Problem Section

This Problem Section is open to everyone; everybody is encouraged to send in solutions and propose problems. Group contributions are welcome. *We no longer reward a book token for solutions, but we will still select the most elegant solutions for publication.* The solutions of the problems in this issue will appear in the next issue.

Problem A (folklore)

Three gamblers each select a non-negative probability distribution with mean 1. Say these distributions are F, G, H. Then x is sampled from F, y is sampled from G, and z is sampled from H. Biggest number wins. What distributions should the gamblers choose?

Problem B (proposed by Hendrik Lenstra)

For given $m \in \mathbb{Z}_{\geq 3}$, consider the regular m-gon inscribed in the unit circle. We denote the surface of this m-gon by A_m. Suppose m is odd. Prove that $2A_m$ and A_{2m} have the same minimal polynomial.

Problem C (proposed by Nicky Hekster)

Let n be a prime number. Show that there are no groups with exactly n elements of order n. What happens with this statement if n is *not* a prime number?

Edition 2018-1 We received solutions from Mohamed Aasila (Strasbourg), Yagub Aliyev (Baku), Bitan Basu (Kolkata), Pieter de Groen (Brussel), Alex Heinis (Amsterdam), Hans van Luipen (Zaltbommel), Abel Putnoki (Eindhoven), Hendrik Reuvers (Maastricht), Hans Samuels Brusse (Den Haag), Seb Schilt (Den Haag), B. Sury (Bengaluru), Araz Yusubov (Baku) and Rob van der Waall (Huizen). The book tokens go to Abel Putnoki, Seb Schilt and Yagub Aliyev.

Problem 2018-1/A

Let f be a function from the set of positive integers to itself such that, for every n, the number of positive integer divisors of n is equal to $f(f(n))$. For example, $f(f(6)) = 4$ and $f(f(25)) = 3$. Prove that if p is prime then $f(p)$ is also prime.

Solution Solved by Mohamed Aasila, Yagub Aliyev, Bitan Basu, Pieter de Groen, Alex Heinis, Hans van Luipen, Abel Putnoki, Hendrik Reuvers, Hans Samuels Brusse, B. Sury, Rob van der Waall.

We follow Abel Putnoki's solution. A prime has two divisors, i.e., $f(f(p)) = 2$. Applying f once more gives $f(f(f(p))) = f(2)$. Hence, the number of divisors of $f(p)$ is equal to $f(2)$. We need to show that $f(2) = 2$.

Since 2 is prime, the number of divisors of $f(2)$ is equal to $f(2)$. The only numbers with this property are 1 and 2. Therefore $f(2)$ is either equal to 1 or 2. We need to rule out 1. Arguing by contradiction, suppose that $f(2) = 1$. Then $f(p)$ has only one divisor, which implies $f(p) = 1$. Applying f once more to $f(2) = 1$ gives $f(1) = f(f(2)) = 2$. Now $f(f(4)) = 3$ is prime and therefore $f(f(f(4))) = 1$. In other words, $f(4)$ has only one divisor, which implies that $f(4) = 1$. This is absurd because if we apply f once more we get $f(1) = f(f(4)) = 3$, contradicting that $f(1) = 2$.

We conclude that the set of primes is invariant under f if $f^2(n)$ is equal to the number of divisors of n. This still leaves us with the question whether such an f exists.

Redacteur: Rob Eggermont
problems@nieuwarchief.nl
www.nieuwarchief.nl/problems

Oplossingen

| Solutions

Problem 2018-1/B

Let n be a positive integer and $F \subseteq 2^{[n]}$ a family of subsets of $[n] = \{1, 2, \ldots, n\}$ that is closed under taking intersections. Suppose that

1. For every $A \in F$ we have: $|A|$ is not divisible by 3.
2. For every pair $i, j \in [n]$ there is an $A \in F$ such that $i, j \in A$.

Show that n is not divisible by 3.

Solution This problem was proposed by Dion Gijswijt, who took it from a recent paper by Martin Nägele, Benny Sudakov and Rico Zenklusen, 'Submodular minimization under congruency constraints', arXiv 1707.06212v2. We received solutions from Hendrik Reuvers, Hans Samuels Brusse and Seb Schilt. Below is the solution by Seb Schilt.

First consider what happens if each $|A|$ is *divisible* by 3. By inclusion-exclusion

$$\left|\bigcup F\right| = \sum_{A \in F} |A| - \sum_{A, A' \in F} |A \cap A'| + \sum_{A, A', A'' \in F} |A \cap A' \cap A'| - \cdots$$

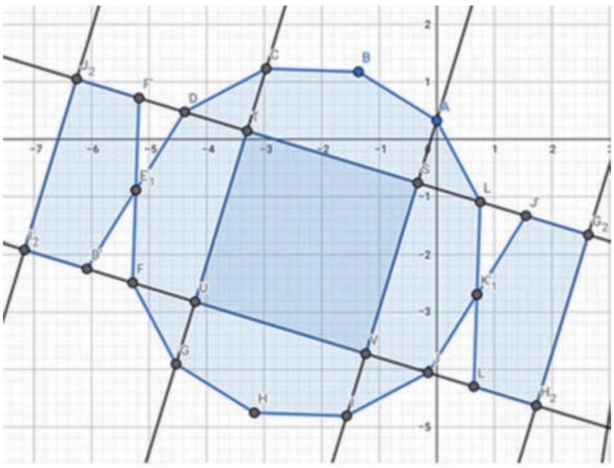
All cardinalities are divisible by 3 since F is intersection-closed, so we have that n is divisible by 3. We only need here that every *element* is in some A, rather than every *pair*. The same idea works if all $|A|$ are 2 modulo 3. Simply add a dummy element to $[n]$ and to all A. If $[n] = \bigcup F$ for an intersection-closed F such that all $|A| \equiv j \bmod k$, then $n \equiv j \bmod k$. We can reduce our problem to this situation by being clever. Let B^2 denote the set of ordered pairs $\{(b_1, b_2); b_1, b_2 \in B\}$. The family F^2 of all A^2 is intersection-closed and contains all elements of $[n]^2$. Now $|A^2| = |A|^2 \equiv 1 \bmod 3$. Therefore, $n^2 \equiv 1 \bmod 3$. In other words, n is not divisible by 3.

What happens if we replace 3 by m in condition 1, and pair by $(m-1)$-tuple in condition 2? Nägele, Sudakov and Zenklusen show that the result remains true if m is a prime power, but it is false for arbitrary m.

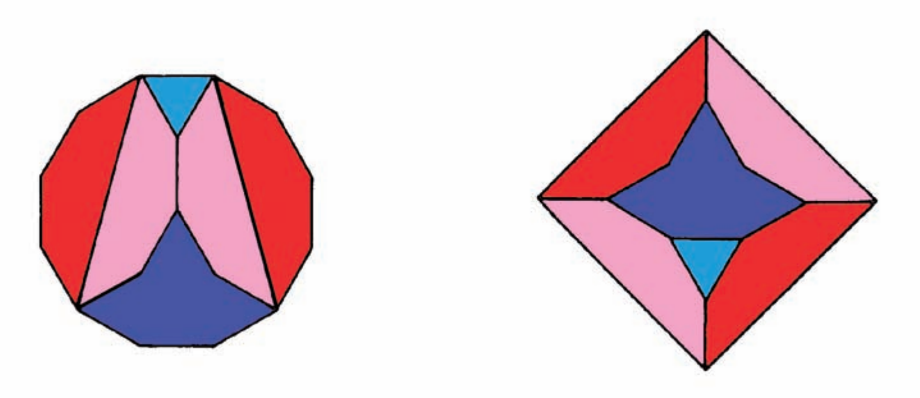
Problem 2018-1/C (proposed by Hendrik Lenstra)

Cut three squares of equal size in exactly the same way into three pieces each in such a way that the resulting nine pieces can be rearranged to form a regular twelve-gon. Open question: Can you cut the three squares into *eight* pieces that form a regular twelve-gon?

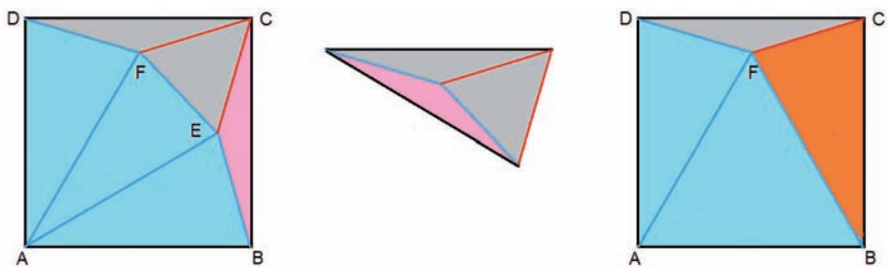
Solution Solved by Yagub Aliyev and Araz Yusubov, Hendrik Reuvers, Hans Samuels Brusse. Yagub Aliyev writes that the entire computer science department of the ADA University in Azerbaijan got interested in this problem and everybody worked on it. Ranging from the secretaries all the way down to the dean. Apart from solving the nine-piece puzzle, they came up with several other interesting ways to divide the twelve-gon into nine pieces that almost assemble into three squares. One of these is illustrated below:



Hendrik Reuvers sends us a dissection of the dodecagon, aka the twelve-gon, from Harry Lindgren's delightful *Recreational Problems in Geometric Dissections*, Dover, 1972. On page 41 it says: "...the chord subtending four sides of a dodecagon is equal to the side of the equivalent square. Knowing this, we can find several dissections merely by trial on the dodecagon, and in almost embarrassing abundance." As illustrated by:



We now follow Hans Samuels Brusse's solution. First observe that the regular twelve-gon has area 3. Therefore, the three squares are unit squares. Think of the twelve-gon as a pie which has twelve triangular pieces. Let $ABCD$ be a square containing one quarter of the twelve-gon with A the circumcentre, B and D are vertices of the twelve-gon. Let E and F be the two intermediate vertices between B and D. Now observe that ABF is equilateral and the isosceles triangles DFC and BEF are congruent. We cut out the quarter of the twelve-gon $ABEFD$ and the remaining part along CF. If we now lay DFC on top of BEF, the two remaining pieces put together form one additional piece of pie. Repeating this for the other two squares gives two additional pieces. These three pieces piece together to form the remaining quarter of the twelve-gon. Piece of cake!



There is a second way to cut the pie. The pieces are $ABFD$, BCF and CDF. Observe that we can combine CDF and $ABFD$ to get $ABEFD$. The remaining piece CDF goes to the fourth quarter. This second solution is actually nicer, since now all pieces are convex.

The problem on eight-pieces-or-less remains open. Finding dissections is difficult. Proving that no dissection exists is even harder. Is it possible to prove that no dissection into eight *convex* pieces exists? Perhaps our friends at the Caspian See would like to try their hands at this.

Edition 2018-2 We received solutions from Pieter de Groen (Brussel), Alexander van Hoorn (Loenersloot), Thijmen Krebs (Nootdorp), Tejaswi Navilarekallu (Bengaluru), Sander Rieken (Arnhem), Hendrik Reuvers (Maastricht), Yan Zhao (Leiden). The book tokens go to Thijmen Krebs, Alexander van Hoorn and Yan Zhao.

Problem 2018-2/A

Let $n > 2$ be an odd integer and let C be an embedding of the circle in $\mathbb{R}^n$. That is, $C = f([0,1])$, where $f : [0,1] \to \mathbb{R}^n$ is continuous, $f(0) = f(1)$, and f is injective on $[0,1)$. Show that there is an affine hyperplane in $\mathbb{R}^n$ that contains at least $n+1$ points from C.

Solution Solved by Thijmen Krebs, Tejaswi Navilarekallu, Sander Rieken, Hendrik Reuvers, Yan Zhao. The idea is that C crosses a hyperplane in an even number of points and that any n points on C are in a hyperplane. Since n is odd, one expects one more crossing. The problem is that not all n points need to be crossings. There can also be turning points. To deal with this, the hyperplane needs to be perturbed. Thijmen Krebs deals with this most succinctly and we follow his solution.

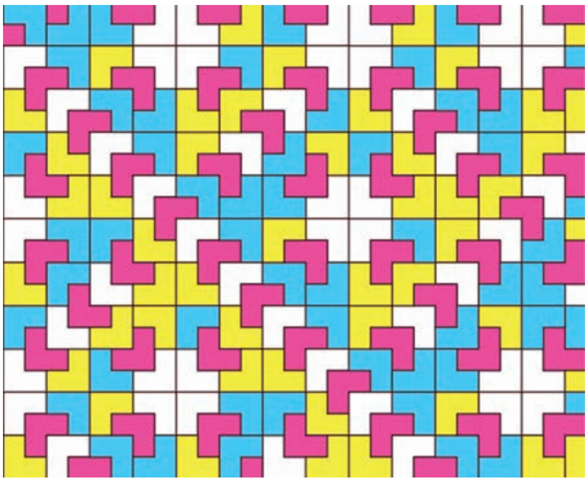
Pick an affine subspace H spanned by any set S of n distinct points from C. If H contains another point from C, or H is not an affine hyperplane, we are done. Since f crosses H an even number of times, it must touch H without crossing in some $f(x) \in S$. That is, there are $0 < a < x < b < 1$ such that $f([a,b])$ lies entirely on one side of H and intersects H only once. By replacing $f(x)$ in the spanning set S with some sufficiently close $f(y) f([a,x])$, we replace S by a set S', deleting a turning point. Getting rid of all the turning points, we eventually find a hyperplane such that the n elements of S are crossings.

Problem 2018-2/B

Place coins on the vertices of the lattice $\mathbb{Z}^2$, all showing heads. You are allowed to flip coins in sets of three at positions $(m,n),(m,n+1)$ and $(m+1,n)$ where m and n can be chosen arbitrarily. Is it possible to achieve a position where two coins are showing tails and all others show heads using finitely many moves?

Solution Solved by Thijmen Krebs, Alexander van Hoorn, Tejaswi Navilarekallu, Hendrik Reuvers, Yan Zhao. The answer is no. Here is Tejaswi Navilarekallu's solution. Suppose we make finitely many moves. Let the choice of the pair (m,n) for each of these moves be $(m_1,n_1),(m_2,n_2),\ldots,(m_k,n_k)$. Without loss of generality, we may assume that $(m_i,n_i) \neq (m_j,n_j)$ for all $1 \le i < j \le k$. Further, we may assume that $m_1 \le m_2 \le \cdots \le m_k$ and if $m_i = m_{i+1}$ for some $1 \le i \le k-1$ then $n_i < n_{i+1}$. The coin at (m_1,n_1) is turned exactly once, so is the coin at (m_k+1,n_k). Let $1 \le i \le k$ be such that $n_i = \max_{1\le j\le k} n_j$. Then the coin at (m_i,n_i+1) is also flipped at most once. Clearly, these three coins are distinct, so there are at least three coins that are showing tails.

Thijmen Krebs says that the positions $(m,n),(m,n+1)$ and $(m+1,n)$ form a *chair*, in analogy with the well-known chair tiling of the plane. In the tiling, chairs can be turned. If we could also do that in our problem, it would have been easy to come up with two coins showing tails in just two moves. Is it possible to come up with a sequence of moves such that only one coin shows tails?



Alexander van Hoorn observes that it is more convenient to describe problem B by polynomials. Let $I \subset \mathbb{F}_2[X,Y]$ be the ideal generated by $1+X+Y$. Does I contain a polynomial of length two, i.e., of the form $X^aY^b + X^cY^d$? The answer is no since modulo I this is $X^a(1+X)^b + X^c(1+X)^d = 0$, which implies $a = c$ and $b = d$.

What if we are allowed to turn the chair upside down? Let J be the ideal generated by $1+X+Y$ and $X+Y+XY$. Does it contain a monomial? The general problem of finding a polynomial of shortest length within an ideal is hard. Harm Derksen and David Masser solved it and their result generalizes the Skolem–Mahler–Lech theorem on linear recurrence. They have written a sequence of papers entitled *Linear equations over multiplicative groups, recurrences, and mixing*, one of which appeared in *Indagationes Mathematicae* in 2015.

Problem 2018-2/C

Say that a natural number is k-repetitive if its decimal expansion is a concatenation of k equal blocks. For instance, 1010 is 2-repetitive and 666 is 3-repetitive. Let R_k be the set of all k-repetitive numbers. Determine its greatest common divisor.

Solution Solved by Pieter de Groen, Thijmen Krebs, Alexander van Hoorn, Tejaswi Navilarekallu, Hendrik Reuvers, Yan Zhao. Problem C is taken from a recent paper by Daniel Kane, Carlo Sanna and Jeffrey Shallit, 'Waring's problem for binary powers', arXiv:1801.04483. Waring's problem is to determine the minimal number $g(k)$ such that every natural number is the sum of $g(k)$ k-th powers. Kane, Sanna and Shallit consider the analogous problem for sums of k-repetitive numbers.
The greatest common divisor is

$$\gcd\left(\frac{10^k-1}{9},k\right)$$

We more or less follow Yan Zhao's solution. Let r_k be the gcd of the set R_k. Consider the set

$$Y_k=\left\{y_a=1+10^a+\cdots+10^{(k-1)a}=\frac{10^{ka}-1}{10^a}:a\in\mathbb{Z}_{>0}\right\}.$$

All elements of R_k are multiples of elements of Y_k, so $\gcd(Y_k)\mid r_k$. Conversely, if a repetitive number has blocks of length a, then it is a multiple of y_a. Clearly, a-digit numbers have gcd equal to one, so $r_k=\gcd(Y_k)$.
Observe that

$$y_a=1+10^{a \bmod k}+\cdots+10^{(k-1)a \bmod k} \pmod{10^k-1}$$

and that y_1 divides 10^k-1. In particular $y_k=k(\bmod y_1)$ which implies that r_k divides $\gcd(y_1,k)$. To finish the proof we need to show that $\min\{\operatorname{ord}_p(y_1),\operatorname{ord}_p(k)\}\le\operatorname{ord}_p(y_a)$ for all primes p. If p does not divide 10^a-1, then $\operatorname{ord}_p(y_a)=\operatorname{ord}_p(10^{ka}-1)\ge\operatorname{ord}_p(y_1)$ since $10^{ka}-1$ is a multiple of y_1. If p does divide 10^a-1, let $10^a=1\,(\bmod p^d)$. By the binomial theorem $10^{ap}=1\,(\bmod p^{d+1})$ and by induction $10^{ap^j}=1\,(\bmod p^{d+j})$. It follows that $\operatorname{ord}_p(y_a)=\operatorname{ord}_p(10^{ka}-1)-\operatorname{ord}_p(10^a-1)\ge\operatorname{ord}_p(k)$. Which finishes the proof. Some of the other solvers point out that the final step in this proof is called the 'lifting-the-exponent lemma'.