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Cuntz–Pimsner cross-pollination

In the last ten years, a class of algebras known as Cuntz–Pimsner algebras has been subject of increasing attention in the operator algebra and noncommutative geometry community, leading many mathematicians to investigate their properties from several perspectives. In search of a unified approach to this rich class of objects, the collaborative workshop *Cuntz–Pimsner Cross-Pollination* has taken place in late June at the Lorentz Center, Leiden, bringing together a diverse group of researchers interested in the many facets of this topic. Francesca Arici, Evgenios Kakariadis and Nadia Larsen report about this event.

Operator algebras provide an elegant setting for many problems in mathematics and physics. They may be configured as algebras of infinite (bounded) matrices and thus they are subject to algebraic and analytic manipulations. This rich structure allows the development of several notions of measurement for the phenomena they encode.

The study of operator algebras is a major undertaking worldwide, continuing to establish connections to physics and to almost every branch of pure mathematics, including dynamical systems, topology, algebra, geometric group theory and number theory. Within this field, two important research directions are currently particularly active: the classification program of C^*-algebras, and Connes' approach to noncommutative geometry (NCG). The workshop revolved around a class of algebras, *Cuntz–Pimsner algebras*, which have proven extremely relevant for both research lines.

Why Cuntz–Pimsner algebras?

Cuntz–Pimsner algebras are C^*-algebras of tractable structure with ample applicability. Their class is broad enough to encompass C^*-algebras of one-dimensional transformations, witnessing many of the most interesting and impressive aspects of C^*-algebra theory. This includes $\mathbb{Z}$-crossed products, such as those used in topological dynamics, algebras associated to Smale spaces [16], and algebras associated to graphs and to symbolic dynamical systems [15]. In physics, connections can be made to gauge theory [8] and quantum mechanics via Connes' noncommutative geometry, where Cuntz–Pimsner algebras serve as models for quantum spaces [3], as well as for topological insulators [7]. Other results indicate the relevance of Cuntz–Pimsner algebras for topics such as C^*-algebraic classification [17, 18, 45] and multiresolution analysis [21, 22].

This broad range of applications has led to the appearance in the literature of many interesting results for sub-classes of Cuntz–Pimsner algebras. As researchers pursued results encompassing their favourite examples, they have contributed different viewpoints to address similar questions. By drawing on this diversity of perspectives, this one week workshop constituted a first step towards building a unified understanding of this class of C^*-algebras as well as to extending their known applications. In order to achieve this aim, the organisers brought together mathematicians from a variety of backgrounds who would contribute to a coordinated and multifaceted analysis of these C^*-algebras.

The workshop

This five day workshop gathered 28 participants from a variety of backgrounds, who also represented an almost 50–50 mix of senior and early career researchers and of male and female mathematicians. The goals of the *Cuntz–Pimsner Cross-Pollination* workshop, as described by the organisers, were to

- extend what is known for specific examples and subclasses into a unified body of knowledge about Cuntz–Pimsner algebras and their higher rank generalisations;
- find new applications of Cuntz–Pimsner algebras;
- strengthen the ties between the different communities of researchers working on Cuntz–Pimsner algebras; and
- highlight and promote the contributions of women researchers to the field of C^*-algebras.

In order to facilitate an exchange of ideas between participants from different backgrounds, the workshop started with four distinguished lectures given by as many experts in the field. These introductory lectures were designed so as to present an overview of the state of the art for topics central to the proposed research program, but which may not have been familiar to all participants.

Twenty years of Cuntz–Pimsner algebras: ***A model for one-dimensional transformations***

Cuntz–Pimsner algebras were first defined two decades ago in Pimsner's seminal paper [41]. The motivation was to unify the C^*-algebraic approach of two important classes of examples in dynamics: the C^*-algebras associated to Markov Chains studied by Cuntz and Krieger [15] in the eighties, and the C^*-algebras coming from a single automorphism. This objective was achieved by constructing universal C^*-algebras associated to an injective C^*-correspondence (that is, a right Hilbert C^*-module with an injective left action of the coefficient C^*-algebra). Almost a decade later Katsura [33] generalised the construction by dropping the injectivity condition, thereby realising even more classes of C^*-algebras: graph algebras, algebras associated to partial automorphisms, and crossed products by C^*-correspondences, to name a few.

Despite being relative newcomers to C^*-algebra theory, many structural properties of Cuntz–Pimsner algebras are well understood, such as exactness and nuclearity [33], simplicity and ideal structure [30, 34], and finiteness and infiniteness [45]. Cuntz–Pimsner algebras also come equipped with useful tools such as gauge-invariant uniqueness theorems and six-term exact sequences in KK-theory, resulting in computable K-theory and K-homology groups. This makes them particularly attractive C^*-algebras and has led to a sharp increase in their use in many branches of C^*-algebra theory. In addition to their crucial role in C^*-algebraic dynamics (cf. [1, 12, 16], see also [31]), Cuntz–Pimsner algebras have recently appeared in the study of quantum groups and quantum spaces [3, 27, 44]. The Cuntz–Pimsner perspective also facilitated both the construction of groupoid models for Kirchberg algebras in [10], and index-theoretic Kasparov products in KK-theory [25]. Furthermore, links with self-similar group actions have been discovered in [20, 38, 39].

Taking motivation from Arveson [6], Fowler [23] established the higher-rank analogue of Toeplitz–Pimsner algebras. This construction quantizes product systems (i.e., a compatible semigroup family of C^*-correspondences) over a C^*-algebra. Sims and Yeend [47] proposed a construction of a Cuntz–Nica–Pimsner C^*-algebra in the context of product systems over quasi-lattice ordered group-subgroup pairs, and Carlsen, Larsen, Sims and Vittadello [13] provided a versatile co-universal counterpart of this C^*-algebra. The new class includes higher-rank graph C^*-algebras, the algebra $\mathcal{Q}_{\mathbb{N}}$ introduced in [11], and crossed products by certain discrete groups and semigroups. Although the higher-rank analogue appeared around the same time with Katsura's work, much less is known about their C^*-structure, as the one-variable tools do not have a direct multi-dimensional translation.

– Wojciech Szymanski from the University of Southern Denmark delivered a lecture on 'Quantum balls, spheres and lens spaces' and explained how a big class of quantum group deformations of classical topological spaces admit various Cuntz–Pimsner models, that reflect different dynamical aspects.
– In his talk 'Classification for non-simple purely infinite C^*-algebras', Ralf Meyer from Göttingen University described how KK-theory, an invariant for C^*-algebras, serves as a powerful tool in the classification program.
– Carla Farsi from University of Colorado Boulder introduced the participants to higher rank analogues of graphs and decompositions of their representations using 'Wavelets and spectral triples for graphs'.
– In her talk about 'KMS states on semigroup C^*-algebras' Nadia Larsen from University of Oslo presented broad classes of motivating examples and obtained general results about this specific class of states.

The first day ended with a *gong session* in which each participant was challenged to describe their current research in two minutes or less. The purpose of the activity was to help participants identify other participants' expertise or interests that might be relevant to their own research, and consequently to inspire discussions. This worked especially well since the gong session naturally continued to a friendly conversation during the wine and cheese party generously offered by the Lorentz Center.

In addition to the introductory lectures, 14 of the workshop participants delivered research talks in a standard conference format during the remainder of the week. The speakers ranged from master students to established researchers; the diversity in seniority created a very welcoming environment, where questions were encouraged.

The talks revolved around recent developments, unpublished work and presentations on:

1. one variable correspondences and the structure of their Cuntz–Pimsner algebras;
2. the structure of their higher rank analogues; and
3. their application in K-theory and Index theory.

1. Baukje Debets (Nijmegen, the Netherlands) described how the structural properties of path groupoids, graphs and C^*-correspondences relate to the C^*-properties of the induced Cuntz–Pimsner algebras. Martijn Caspers (Delft, the Netherlands) showed that arbitrary free quantum groups are strongly solid in the sense of Ozawa and Popa by using a W^*-correspondence introduced by Cipriani and Sauvageot. Motivated by the passage from the Toeplitz–Pimsner to the Cuntz–Pimsner algebras in the groupoid picture of Yeend, Gilles de Castro (Florianópolis, Brazil) showed how pointless topology can be used to recover the boundary path space of a topological graph. The groupoid theory has also been the source of motivation for constructing minimal homeomorphisms on CW-complexes with prescribed K-theory, presented by Robin Deeley (Boulder, United States). A wide subclass of Cuntz–Pimsner algebras can be constructed from labeled directed graphs and Ja A Jeong (Seoul, Korea) discussed their ideal structure and sufficient conditions for being AF and purely infinite. Inspired by earlier work of Muhly–Tomforde on graph C^*-algebras, Bartosz Kwasniewski (Białystok, Poland) discussed topological freeness and induced criteria of simplicity for general Cuntz–Pimsner algebras.

2. Camila Fabre Sehnem (Göttingen, Germany) showed how the theory of one-variable (relative) Cuntz–Pimsner algebras can be used to show amenability of Fell bundles, a large class that engulfs product system constructions. James Fletcher (Wellington, New Zealand) showed how the Cuntz–Nica–Pimsner algebra of locally convex higher-rank graphs can be realized through iterations of the monochromatic Cuntz–Pimsner algebras. A description of the Cuntz–Nica–Pimsner representations for the class of strong compactly aligned $\mathbb{Z}_+^N$-product systems as the higher-rank anlogue of Katsura's ideal was shown by Evgenios Kakariadis (Newcastle-upon-Tyne, United Kingdom). Sergey Neshveyev (Oslo, Norway) presented the full description of the finite type KMS-states for product systems of rather general quasi-lattices, encompassing a great number of previous results. More information on the KMS-simplex of right LCM semigroup C^*-algebras was given by Nicolai Stammeier (Oslo, Norway) as well as potential connections with other structural data.

3. Anna Duwenig (Victoria BC, Canada) discussed how to expand on Connes's proof of Poincaré self-duality of the irrational rotation algebra by giving a geometric description of the dual class. Marzieh Forough (Tehran, Iran) considered finite group actions and variants of the Rohklin property to address questions of quasidiagonality and Cuntz–Pimsner algebras. Bram Mesland (Bonn, Germany) described the groupoid model for the bulk-boundary correspondence in solid state physics, based on the notion of Delaunay sets. Quite remarkably, in the one dimensional case, this groupoid C^*-algebra can be described as a Cuntz–Pimsner algebra and the extension class agrees with the class of the bulk-boundary extension.

To further stimulate interaction between the participants, the third day of the workshop included a *plenary discussion session*. Topics of discussion were suggested by the participants, and consisted of problems which they viewed as central open questions about Cuntz–Pimsner algebras and their generalisations. On the final day, the workshop concluded with another discussion, stemming from problems presented earlier in the workshop, about a future research agenda for the continued study of Cuntz–Pimsner algebras and their applications.

Committees

The scientific organisers of the meeting were: Francesca Arici (Leipzig, Germany), Sara Arklint (Copenhagen, Denmark), Elizabeth Gillaspy (Missoula, USA), Karen Strung

Quantum spaces: *From dynamical systems to algebraic deformations and back*

Several results indicate the potential of the Cuntz–Pimsner perspective to improve our understanding of quantum principal bundles and quantum homogeneous spaces. In [27,28], Hong and Szymański show that some well-known examples of quantum spaces — such as quantum spheres, quantum lens spaces and quantum projective spaces — can be realised as graph C^*-algebras, and hence as Cuntz–Pimsner algebras. A complementary approach was taken in [3], where the authors constructed a Cuntz–Pimsner model for the Vaskman–Soibelman quantum sphere and lens spaces as principal circle bundle over quantum (possibly weighted) projective spaces. These two models encode different data: for example, the graph algebra perspective enables explicit computation of the K-theory and primitive ideal space, while the approach of [3] gives a noncommutative Gysin sequence. Quite surprisingly, some of these algebras, such as quantum weighted projective spaces, admit a quantum principal bundle structure when no classical analogue exists [9].

While spectral obstructions imply that, unlike those in [27,28], not all quantum homogeneous spaces will have graph algebra models, the question whether a given quantum space may admit a Cuntz–Pimsner model still stands open. There is some evidence that this problem may have a positive answer. First of all, any C^*-algebra endowed with a semi-saturated torus action admits a product system Cuntz–Pimsner model, which means that they can be obtained as iterated Cuntz–Pimsner algebras. Moreover, work in progress by various authors using generalisation of C^*-correspondences like product and subproduct systems [46,50], shows that the Cuntz–Pimsner framework has the potential of extending to quantum bundles with different structure, like quantum flag manifolds and sphere bundles. There is evidence that a combination of C^*-correspondences with subproduct systems can provide a rigid analysis of the ambient structures, e.g., for monomial ideals [32]. Results on the K-theoretic range of these algebras, such as those in [5], may help with the investigation.

KK-theory, wavelets and multi resolution analysis: ***Bridging geometry and analysis***

Connes' approach to noncommutative geometry [14] finds one of its biggest sources of inspiration in the celebrated Atiyah–Singer index theorem (see [36] for a popular overview of the topic), whose greatest merit is that of establishing a bridge between topology on one side, and the analysis of differential operators on the other. Similarly, in NCG, a central notion is that of a *spectral triple* $(A,\mathcal{H},D)$, which consists of an involutive algebra A of operators acting on a Hilbert space $\mathcal{H}$ and a self-adjoint operator D on the same Hilbert space, subject to certain conditions. Spectral triples are modelled on the classical geometric notion of Riemannian spin manifold, with the algebra A playing the role of the algebra of smooth functions $C^\infty(M)$ on a manifold. One can also view a spectral triple $(A,\mathcal{H},D)$ as an unbounded Kasparov module representing a class in the bivariant K-theory group $KK^1(A,\mathbb{C})$. For a friendly introduction to index theory, Kasparov's bivariant K-theory, and their applications to mathematical physics, we refer the readers to the recent conference report [4].

Every Cuntz–Pimsner algebra O_E associated to a Hilbert bimodule E over A gives rise to an element of $KK^1(O_E,A)$ via the defining extension. Work by Goffeng, Mesland and Rennie [25] has established that many of the spectral triples constructed in recent years [24, 43] arise as the localisation of precisely such a Cuntz–Pimsner Kasparov module. In the setting of graph and k-graph algebras, Farsi, Gillaspy, Julien, Kang and Packer [21, 22] have studied spectral triples via their relationship to wavelets and multiresolution analysis. Wavelets use geometric information (in the form of dilation and translation operators) to generate an orthonormal basis for the Hilbert space on which the algebra is represented. The investigation of the relation between wavelets on fractals and Cuntz–Krieger algebras, a special class of Cuntz–Pimsner algebras, was initiated in [37]. Uniting the Kasparov module and the multiresolution analysis approach, by constructing wavelet decompositions in the context of general Cuntz–Pimsner algebras, has the potential of making index pairings easier to compute, while at the same time shedding new light into the field of time-frequency analysis.

Kubo–Martin–Schwinger states: ***Invariants for the dynamics on Pimsner algebras***

The KMS$_\beta$ states of a C^*-dynamical system are the equilibrium states of the system at 'inverse temperature' $\beta \in \mathbb{R}$. As the parameter β varies, changes of the KMS-simplex provide useful and occasionally unexpected information about the structure of the C^*-dynamical system. The preferred dynamics for Pimsner algebras come from the gauge action, and the classification of the β's where phase transitions occur can be used for classification up to equivariant isomorphisms.

Beginning with the work of Laca and Neshveyev [35] and of Pinzari, Watatani and Yonetani [42], the KMS states of many subclasses of Pimsner algebras have been extensively studied. It appears that the Toeplitz–Pimsner algebra admits a rich structure coming from the KMS states induced by the generalized compacts (finite type) and the KMS states that factor through the Cuntz–Pimsner algebra (infinite type). Consequently there are two critical inverse temperatures: one above which there are no KMS states of infinite type and one below which there are no KMS states at all. Remarkably these values can be connected to the structural data of the C^*-correspondence. For example in the case of irreducible graphs they both coincide with the logarithm of the Perron–Frobenius eigenvalue of the adjacency matrix [29]. This has been a recurrent theme in the past 15 years for specific examples of C^*-correspondences, in particular when they admit finite Parseval frames. Until recently the literature contained only partial results about the KMS states for the higher rank analogues (mainly focused on finite type) due to Hong, Larsen and Szymański [26], and Afsar, an Huef and Raeburn [2]. Nevertheless in recent months there have been massive developments on that front. The workshop has provided an excellent opportunity for the participants to discuss announced and on-going work of Christensen (higher-rank finite graphs), Kakariadis ($\mathbb{Z}_+^N$-product systems with finite Parseval frames), and Afsar, Larsen and Neshveyev (general product systems). It has been evidenced that the parametrisation of the KMS-simplices and the computation of the critical values is tractable for a wide class of Pimsner algebras, both in the case of a single correspondence and in higher-rank cases. Work of Neshveyev [40] and Thomsen [48] also indicates that generalizing the gauge action on the Cuntz–Pimsner algebra may lead to intriguing KMS structures. In the k-graph setting, this question is the focus of current work by Farsi, Gillaspy, Larsen and Packer.

(Nijmegen, Netherlands). The organising committee, consisting of early career researchers, was accompanied by an advisory committee: Astrid an Huef (Wellington, New Zealand), Takeshi Katsura (Tokio, Japan), Sergey Neshveyev (Oslo, Norway), Wojciech Szymanski (Odense, Denmark). One of the most important tasks faced by the committees was to preserve a healthy gender ratio and ensure diversity in terms of geography and seniority. The workshop was held in cooperation with the Association for Women in Mathematics (AWM) and supported their 'Welcoming Environment Statement' [51].

Demographics

The organizing committee comprised four women. Women made up 46% of the participants in the workshop, and more specifically 50% of the plenary speakers and 36% of the contributed talks. These are comparatively high figures for a workshop in the field of operator algebras, where generally women have been severely underrepresented. The organisers hope that these numbers will serve as an inspiration for similar events in the future.

Participants attended from a wide range of geographical locations, including Brazil, Canada, Iran, Japan, Korea, New Zealand, the United States, and of course a number of European countries including the Neth-

C^*-algebraic classification: *Understanding C^*-algebras from their group theoretic imprint*

The classification program for simple amenable C^*-algebras is a far reaching endeavor in the theory of operator algebras which has seen spectacular fulfillment in the past couple of years. Building on work of Glimm, Bratteli and Elliott in the 1960's and 1970's, Elliott conjectured that separable, simple, amenable C^*-algebras can be classified by their K-theoretic invariants. The first major successes were seen in the early 1990's, culminating with the Kirchberg–Phillips result on the classification of separable, simple, purely infinite, amenable C^*-algebras.

Outstanding work by many hands has lead to a resounding success of Elliott's classification program in the simple case, see [19,49] and the references therein. Thus it is now known that all simple separable unital C^*-algebras with finite nuclear dimension satisfying the UCT can be distinguished by their Elliott invariant (consisting of K-theoretic and tracial data). The final piece of the puzzle is the remarkable result of Tikuisis, White and Winter [49] that settles quasidiagonality of faithful traces on separable nuclear C^*-algebras which satisfy the UCT.

Meanwhile, the unital graph algebras — with no assumption of simplicity(!) — have also been classified, both by a K-theoretic invariant, and also by means of geometric data associated to the graphs [18].

Although at present it is unreasonable to suggest that there is an obtainable classification theorem for all Cuntz–Pimsner algebras, one particular subclass appears manageable: Cuntz–Pimsner algebras associated to quantum bundles always have finitely many primitive ideals, are in the UCT class, and, for those that have graph algebra models, already fit under the theorem of [18]. Furthermore, their K-theory is known in many cases, making this class particularly amenable to classification. During the workshop there was a focus on the study of this particular class.

erlands. There was an almost equal split between Senior Researchers, Early Career Researchers with tenured/tenure-track positions, postdoctoral fellows and postgraduate students. By inviting a diverse pool of participants, the organisers succeeded in creating a welcoming and inclusive atmosphere during the workshop, which facilitated discussion and scientific collaboration. That was of extreme value for stimulating discussions especially after the gong session and during the open problem session. It should perhaps be stressed that the very diversity of the participants' expertise and interests was a crucial reason why so many women were present at the workshop.

Participant feedback

Several of the workshop participants reported to the organisers that the structure of the meeting facilitated conversations with other researchers, including researchers with whom they had not previously interacted. It was also observed that the atmosphere of the workshop felt more inclusive and supportive than other conferences due to its collaborative atmosphere, relatively small size and diversity of participants.

Finally, participants expressed enthusiasm for the idea of holding a follow-up conference in approximately two years' time, to continue to strengthen the research community centered on Cuntz–Pimsner algebras.

One cannot underestimate the excellent facilities and professionalism of organisation provided by the hosts at the Lorentz Center towards the success of the meeting. Everything worked perfectly. The facilities at the center, with boards, various sitting areas and glass walls make for easy interaction between participants. The relatively small scale of the meeting means that all participants get to talk to essentially everybody else. As a result, several new collaborations and research projects are expected to blossom through this *cross-pollination*.

Participants from left to right. Standing: Baukje Debets, Gilles de Castro, Ralf Meyer, Nadia Larsen, Walter van Suijlekom, Ja A Jeong, Gi Hyun Park, Bram Mesland, Astrid an Huef, Nicolai Stammeier, Elizabeth Gillaspy, Robin Deeley, Camila Fabre Sehnem, Bartosz Kwasniewski, Sophie Emma Mikkelsen, Takeshi Katsura, Carla Farsi, Marzieh Forough. Sitting: James Fletcher, Evgenios Kakariadis, Sergey Neshveyev, Marcelo Laca, Francesca Arici, Karen Strung, Wojciech Szymanski. Not in the picture: Martijn Caspers, Anna Duwenig and Chiara Pagani.

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