Problem

| Problem Section

This Problem Section is open to everyone; everybody is encouraged to send in solutions and propose problems. Group contributions are welcome. For each problem, the most elegant correct solution will be rewarded with a book token worth € 20. (To compete for the book token you should have a postal address in The Netherlands.)
Please send your submission by e-mail (LaTeX is preferred), including your name and address to problems@nieuwarchief.nl.
The deadline for solutions to the problems in this edition is 1 December 2017.

Problem A

Let n be a natural number and suppose that $A_1,\dots,A_n$ are different subsets of $\{1,\dots,n\}$. Prove that there is a $k \in \{1,\dots,n\}$ such that $A_1 \setminus \{k\},\dots,A_n \setminus \{k\}$ are different.

Problem B (proposed by Hans Zantema)

Let $f,g:\mathbb{N} \to \mathbb{N}$ be strictly increasing functions. Prove that there exists an $n \in \mathbb{N}$ such that $f(g(g(n))) \geq g(f(n))$.

Problem C (proposed by René Pannekoek)

Determine all $n \in \mathbb{N}$ such that $2^n - 1$ divides $3^n - 1$.

Edition 2017-1 We received solutions from Pieter de Groen (Brussel), Alex Heinis (Amsterdam), Thijmen Krebs (Nootdorp), Hendrik Reuvers (Maastricht), Hans Samuels Brusse (Den Haag), Djurre Tijsma (Zeist), Rob van der Waall (Huizen), Martijn Weterings (Sion) and Hans Zantema (Eindhoven). The book tokens for problems A, B and C go to Rob van der Waall, Hans Samuels Brusse, respectively Hans Zantema.

Problem 2017-1/A

Reconstruction. Suppose you get to know the n midpoints of the n edges of a polygon. Can you determine the polygon?

Solution Solved by Pieter de Groen, Thijmen Krebs, Hendrik Reuvers, Hans Samuels Brusse, Rob van der Waall, Martijn Weterings and Hans Zantema. Almost all solutions are similar. Rob van der Waall presents a purely geometric construction.
No if n is even. Yes if n is odd. Denote the midpoints by $m_1,\dots,m_n$. If n is odd, then the alternating sum $m_1 - m_2 + \cdots + m_n$ is equal to the polygon's initial vertex. All vertices can be reconstructed like this. If n is even, then we may add an arbitrary v to the even vertices and subtract it from the odd vertices without changing the midpoints. We may place the initial vertex at an arbitrary point, and we can construct a polygon by reflections in the midpoints.

Problem 2017-1/B

Large is odd. Let G be a graph with vertices V and edges E. A subset $U \subset V$ is called *large* if every vertex that is not in U has a neighbor in U. Prove that the number of large subsets is odd.

Solution We received solutions from Hans Samuels Brusse and Pieter de Groen. Thijmen Krebs noticed that this is a result of Brouwer, Csorba and Schrijver, which can be found

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through www.win.tue.nl/aeb/preprints/domin4a.pdf. Hans Samuels Brusse uses induction on the number of vertices. Pieter de Groen uses induction on the number of edges. The wonderful solution below is due to Brouwer, Csorba and Schrijver.

For every $U \subset V$, let $\bar{U}$ be the subset of vertices in $V \setminus U$ that contain no neighbor in U. In particular, U is large if and only if $\bar{U}$ is empty. Consider the family $\mathcal{F}$ of all pairs (U,W) such that $W \subset \bar{U}$. For a fixed U, there are $2^{|\bar{U}|}$ such pairs. In particular, for a fixed U, there are an odd number of such pairs if and only if U is large. Therefore, the parity of the number of large sets is equal to the parity of $|\mathcal{F}|$. This family is invariant under the involution $(U,W) \mapsto (W,U)$, which has $(\emptyset,\emptyset)$ as a single fixed point. Therefore, the parity of $|\mathcal{F}|$ is odd.

Problem 2017-1/C

Meanders. Let n be an even number. Consider the integers 1 to n in the complex plane and connect them by semicircles centered around $\frac{n+1}{2}$ in the upper half plane. Let $n = a + b$ for even numbers a and b. Connect the first a integers by semicircles around $\frac{a+1}{2}$ in the lower half plane. Similarly, connect the last b integers by semicircles around $\frac{a+b+1}{2}$ in the lower half plane. The resulting curve, or set of curves, is a meander. For which a and b is it connected?

Solution We received solutions from Pieter de Groen, Alex Heinis, Thijmen Krebs, Hendrik Reuvers, Hans Samuels Brusse, Djurre Tijsma and Hans Zantema. All solutions are similar. Alex Heinis remarks that this problem is related to the combinatorics of Hedlund words, as studied in his thesis. Indeed, there is more to this problem. Here we look at meanders for a sum of two even numbers, but the definition extends to sums of more than two numbers. For sums of four or more even numbers it is much harder to decide if the meander is connected. The interested reader should watch Maryam Mirzakhani's Marston Morse lecture https://www.youtube.com/watch?v=mxPE6vYwqLg.

The semicircles connect an even number to an odd number. If we start from an arbitrary even number and we trace an upper semicircle followed by a lower semicircle, then we are back at an even number. More specifically, if we start from $2k$, then we are back at $2k + a$ modulo n. The meander is connected if and only if it takes $n/2$ iterations before we are back at $2k$. In other words, the least common multiple of a and n is equal to $na/2$. It is easier to say that the greatest common divisor of a and n is equal to 2. Since $n = a + b$, the meander is connected if and only if $\gcd(a,b) = 2$.