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Education

The use of games and their history to improve secondary school students' skills in mathematics reasoning and problem solving

This article resulted from the talk 'Machines designed to play Nim games: a possible (re) use in the modern mathematics curriculum?', presented by Lisa Rougetet at the Wintersymposium of the KWG in January 2017. This talk, and the underlying studies, have been motivated by the latest reform (September 2016) of the French high school education system, which has led to changes in the curricula.

In mathematics, a new theme entitled 'algorithmics and programming' aims at initiating pupils (7th–9th grades) to 'write, develop and run a simple program'. To achieve this, the curriculum offers several class activities centered on 'games in a maze, [...], Nim game and Tic-Tac-Toe'. Of course, these examples are only suggestions, and teachers are free to approach the new theme with other activities. In this article, we would like to explain how Nim game, and some variants, could enable secondary school students to investigate problems through a stimulating experimental approach that helps to develop mathematical reasoning. As the idea of introducing games in class to tackle algorithmics notions is brand new (and as it is just a suggestion for teachers), we don't have many concrete teaching outcomes to present. However, we report on experiments that had been led with pupils in out-of-school contexts, using Nim game.

The resources provided by the French government to set up the new 'algorithmics and programming' theme point out that pupils should develop algorithmic thinking. The expected skills at the end of the 9th grade are:

- breaking down a problem: analyzing a complex problem, and dividing it into sub problems;
- recognizing patterns: looking for invariants and repetition;
- generalization and abstraction;
- designing algorithms.

To do so, one of the proposed activities, among others, is to program playful applications (mazes, pong, battleship, Nim, Tic-Tac-Toe). This idea raised the question of using Nim-like games[1] in mathematics teaching; what skills do they help to develop, could they be used in class to help pupils to acquire algorithmic thinking, or more generally, mathematical reasoning? To try to answer these questions, the present article will focus on the Nim game, which is one of the most famous combinatorial games. We will see, through the Nim game, that combinatorial games may play a role in learning the know-how of mathematical activities[2] and that they offer research activities (in which students have to think for themselves and are, in a way, placed in the situation the mathematician faced when he was looking for the solution of the game for the first time) for students from high school to university. Knowing their history and the development of their mathematical theory is important to understand the epistemic approach that students face when they play the game and try to solve it.

Nim game rules and solution

The Nim game — as it was introduced for the first time in 1901 — is a take away game: usually three (or more) rows of matches are set on a table; each row contains a different number of matches. Alternately, both players select one of the rows and remove as many matches as they want: one, two, ..., or even the whole row. The player who takes the last match(es) wins the game. An example of a possible initial position is shown in Figure 1. There are many different versions of Nim, called Nim-like games, depending on the initial number of rows and matches, and the rules according to which they can be removed.[3]

The strategy to win at Nim is as follows: first, for each row, the number of matches must be written in binary. Then, these binary numbers are placed in three horizontal lines so that the units are in the same vertical column. The sum of each column is calculated and if all of them are congruent to 0 mod 2, the position is called a *losing position*. Such positions should be reached at each move in order to win the game, because the opponent cannot reach another losing position in the next move.[4] The general theory of Nim using the binary system can hardly be discovered by pupils, but the aforesaid properties of winning and losing positions, defined in a recursive manner, are often understood rather well, even if not totally assimilated.[5]

History of Nim and combinatorial game theory

The starting point of the history of combinatorial game theory is commonly dated to 1901, when the Nim game was first mentioned under this name in an article published in the *Annals of Mathematics* by Charles Leonard Bouton (1869–1922), a mathematician from Harvard.[6] In this article, Bouton gives the complete mathematical solution to Nim. Bouton's article is considered as a cornerstone of the development of combinatorial game theory, because, unlike recreational mathematics books of the sixteenth, seventeenth and eighteenth centuries, it gives a solution to any possible initial position, no matter the number of rows and objects in each row.

After Bouton's article, combinatorial game theory was developed by other mathematicians and became a beautiful abstraction with John Conway's surreal numbers theory in 1976. Thanks to Conway's work, combinatorial games became a separate branch of mathematics and computer science. Nowadays, combinatorial game theory is a branch of mathematics, which links up mathematics (graph theory, set theory), computer science (game programming, artificial intelligence) and education.

Being familiar with the history and the mathematical development of Nim-like games has a double interest: on one hand, it provides material to set up interdisciplinary activities (required in the French new curriculum) that combine history, mathematics and computer science. On the other hand, playing Nim-like games in order to solve them (finding a winning strategy) places students in an epistemic approach through a singular experimental situation in which the required knowledge will appear as the optimal solution to the problem.[7] Looking for the best moves to play, in order to win the game at the end, coincides with the knowledge we want the pupils to acquire, i.e. winning and losing position, strategy, backward reasoning, et cetera (unlike other games that require the use of multiplication tables, for instance).

Didactical aspects of Nim-like games

Different kinds of knowledge are associated to combinatorial games[8]: non-institutional knowledge, which includes specific notions of combinatorial games such as winning and losing position, and winning strategies, and institutional knowledge, which consists of mathematical notions that can be found in the usual curriculum, for example integers properties, induction, recursion, et cetera. The latter aspect shows that the use of combinatorial games in class is appropriate for learning fundamental notions in mathematics through a playful situation.[9] Playing and analyzing Nim-like games are also appropriate to develop algorithmic skills. First, to understand how a game can be won in a general case, one needs to analyze simpler positions, i.e. positions that occur at the end of the game. For instance, when they play Nim, pupils quickly start to analyze whether $(1,1)$, $(1,2)$ or $(1,1,1)$ are winning or losing positions. This requires to break down the initial problem, which is quite complex, into simpler sub problems. Then, they should try to generalize some configurations. For instance, once pupils have understood that $(1,1)$ and $(2,2)$ are losing positions, they can figure out that (n,n) is also a losing position, for any n. The same occurs with $(1,1,n)$, which is a winning position for any n. These observations have been made on a group of twelve pupils (14–15 years old) in the context of a 'mathematical summer camp' organized by 'Plaisir Maths' in June 2016.[10] Then, even if games differ in their design from one to another, the mechanisms implemented to develop winning strategies are the same, centered on the properties of winning and losing positions. Thus, they stimulate pattern recognition (to identify a configuration seen before) and abstraction (to understand the underlying strategy regardless of the form of the game).

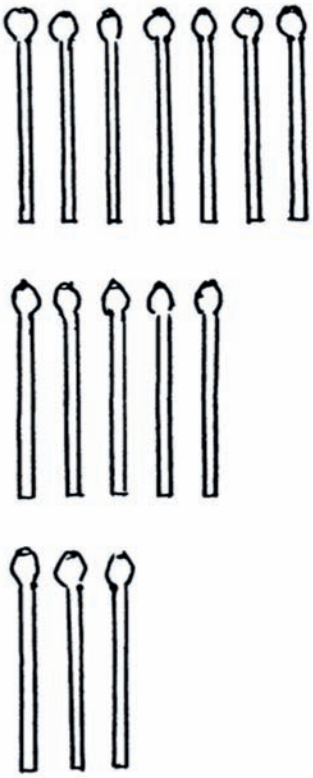

Figure 1 Example of an initial position of the Nim game: three rows containing respectively 7, 5 and 3 matches.

Beyond these notions connected to combinatorial games properties, games such as Nim could be a good training to practice mental arithmetic and to develop abstract thinking (e.g. speculating about what the opponent will play to better counter him). They could be used to introduce enumeration (given a position, what are the possible moves?) and to illustrate graph theory through game trees, which represent the possible connections between the positions in the game. But, as we mentioned earlier, since the reform of the curriculum is brand new, the use of combinatorial games in class is still in its early stages, and we have no records on the actual teaching outcomes. Furthermore, teachers first need to train themselves on Nim game, its strategy, its theory (less known than Tic-Tac-Toe for example), and they can attend study days devoted on this kind of training (but not mandatory).

Furthermore, combinatorial games enable the creation of situations in which students tackle problems with an autonomous research activity through an experimental approach that combine three kinds of actions in problem solving: presenting new problems, experimenting–observing–confirming, and trying to prove the correctness of an approach.[11] Combinatorial games perfectly constitute a mathematical activity, as they allow for many fundamental aspects of mathematical activities: discovery phases, conjectures, trial and error, reformulation, proof arguments, et cetera. It is interesting to submit a given position of a game to pupils and to ask them to determine its nature (winning or losing). They can put their ideas to the test and confirm or invalidate their hypothesis by playing directly.[12]

Besides, as games are often considered more enjoyable than plain mathematical tasks, pupils may be more focused on finding the winning strategy. Using counters in Nim-like games can help, for instance, to better picture the problem and enables a quick representation of a position at any moment of the activity. Action is an essential part of mathematics learning for it gives meaning to the mathematical activity through its experimental dimension.

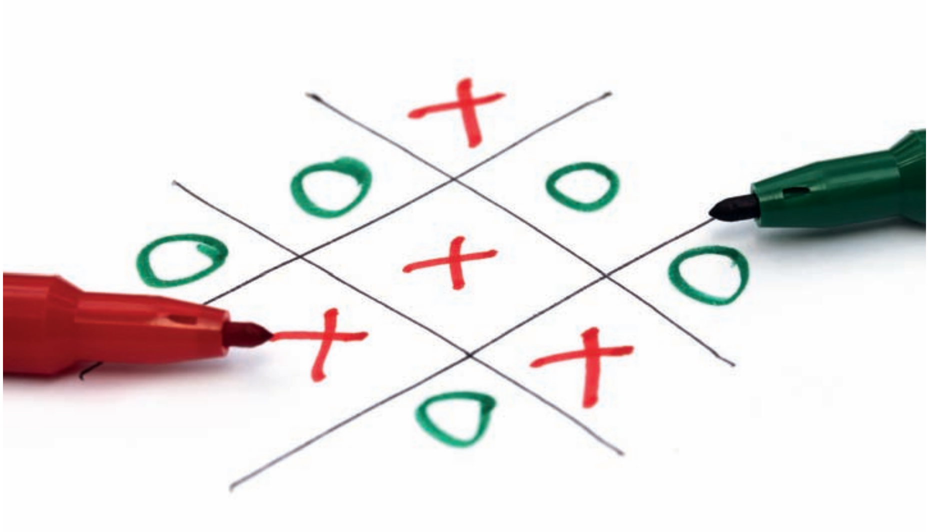

Figure 2 Tic-Tac-Toe is an example of a combinatorial game.

Conclusion

Combinatorial games are well-suited for active learning, and improve pupils' personal relationship with mathematics. In this kind of playful activity, pupils' involvement is different, because it stimulates group work and communication, avoids them from getting discouraged, and can motivate pupils with learning difficulties: wanting to find how to win is a strong pedagogical lever. The game in itself is the environment the pupils act on and with, and the fact that it is fun helps them to learn mathematics. We are confident that the strength of games for teaching makes them essential tools for mathematics education, because of the features they share with learning situations in which the student is involved; then he becomes actor of his learning and accepts the responsibility of the knowledge transfer.

Notes

1 In general, *combinatorial* games can be used. In a combinatorial game, there are two players, playing alternately. Usually, there are a finite number of positions and the information is complete — which means both players know what is going on at any moment of the game. There are no chance moves such as rolling dice or shuffling card and generally the player who finds himself unable to play loses. *Tic-Tac-Toe*, *Connect Four* and *Checkers* are examples of combinatorial games. In a combinatorial game there is a strategy — a sequence of moves — that enables one player to win the game whatever his opponent plays. It means that, theoretically, it is possible to know the nature of any arbitrary position in the game (*losing* or *winning*) if we assume both players play optimally (in practice, the analysis is more complex, because of the high number of possible positions in most games).

2 With 'know-how' we mean the knowledge, methods and techniques that are the base of all mathematical activity, such as experimentations, doing a particular case study, modeling, construction of proofs and definitions, et cetera. See details in: X. Colipan, *Étude didactique des situations de recherche pour la classe concernant des jeux combinatoires de type Nim*, Doctoral thesis, Institut Fourier, Grenoble, France, 2014. Retrieved from https://tel.archives-ouvertes.fr/tel-01121726/.

3 For instance, few years after Bouton, a Dutch mathematician named Willem Abraham Wythoff (1865–1939) published an article in the *Nieuw Archief voor Wiskunde*, presenting the following version of Nim: there are only two rows of matches, and at each turn players can remove any number of matches from one row only OR remove the same number of matches in both rows.

4 That is why the position is called a *losing position*: there exists a winning strategy for the player who has just played, and not for the next player to play. A position is a *winning position* if there exists a winning strategy for the next player to play.

5 For further details, see X. Colipan, 2014, p. 140, in the context of the geometrical Euclidean game.

6 Actually, things are not so definite: we have found earlier analyzes of combinatorial games in recreational mathematics books from the sixteenth century and thereafter. See, e.g. L. Rougetet, A Prehistory of Nim, in: M. Pitici (ed.), *Best Writing on Mathematics 2015*, Princeton University Press, 2016, pp. 207-214.

7 G. Brousseau, *Théorie des situations didactiques*, La Pensée Sauvage, 1998, p. 49.

8 X. Colipan, 2014, p. 36.

9 This approach is not new: the oldest analyzes of games — which we would qualify nowadays as combinatorial — have been found in recreational mathematics books in the sixteenth century. Their main purpose was to tickle curiosity, but also to use a playful dimension to obtain a mathematical result.

10 'Plaisir Maths' is a structure of mathematics popularization, which gather animators, teachers and researchers to create and organize playful and didactical mathematical projects.

11 N. Giroud, *Étude de la démarche expérimentale pour les situations de recherche pour la classe*, Doctoral thesis, Institut Fourier, Grenoble, France, 2011. Retrieved from https://tel.archives-ouvertes.fr/tel-00649159, p. 7.

12 As far as I know, Colipan and the group she worked with during her Ph.D. are the only ones, in France, who experimented combinatorial games in class and published didactical results on it. The federative structure 'Maths à Modeler' (whose aim is to propose workshops to the general public to discover fundamental computer sciences and mathematics) and 'Plaisir Maths' also use combinatorial games in their actions of scientific dissemination.