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Book Review Bertrand Duplantier and Vincent Rivasseau (eds.): Henri Poincaré, 1912–2012

A global view of Poincaré's scientific work

On the occasion of the centennial of the death of Henri Poincaré in 1912 the monograph *Henri Poincaré, 1912–2012* appeared, edited by Bertrand Duplantier and Vincent Rivasseau. This book is published as the thirteenth volume of the Poincaré Seminar Series. Ferdinand Verhulst, author of *Henri Poincaré, impatient genius*, discusses each of the four chapters of the book.

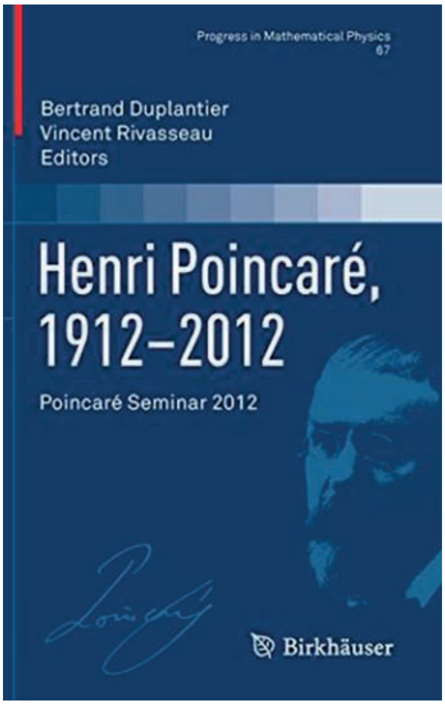
In a long period around 1900 two giants of mathematics dominated the scene: David Hilbert and Henri Poincaré. It is nearly impossible to compare them, Hilbert who solved essential mathematical problems and who created fundamental generalizations of concepts, Poincaré who invented many new concepts to solve classical problems and who also made important contributions to physics and engineering. While Hilbert laid solid foundations of theory we are using to build on, the reaction to Poincaré has been completely different. We see each year new papers and books about Poincaré's work, he was a fountain of ideas and theories. As Vito Volterra said about him: "He was the impressionist among the mathematicians."

This book is the thirteenth in a series of Poincaré seminars aimed at a wide public of scientists. It consists of four chapters and a description of a movie focusing on scientific topics and experiments. To some extent it aims at presenting a global view of Poincaré's scientific work; we will discuss the chapters separately.

Poincaré's optics

The first chapter is written by Olivier Darrigol (pp. 1–50). Interestingly and unusually, Darrigol is a historian of science who seems to know and understand the mathematical physics of around 1900. In an interesting fifty pages he describes how in Poincaré's mathematical optics this leads Poincaré to diffraction, electrodynamics and relativity. Physics is intimately connected in this work with geometry. To avoid confusion to the reader: this connection does not reduce physics to mathematics, that would not be fruitful but mathematics is an efficient and very useful tool to describe the physical phenomena. Already much earlier, for Fresnel the concept of the ether was not necessary, but it often came up because of the inspiration to develop optical theory derived from the theory of elasticity where waves do need a medium. Maxwell unified optics and electromagnetism but the French did not like his results because of his non-axiomatic style. Poincaré however, admired Maxwell's results while deploring his lack of rigorous mathematical physics; the French took classical mechanics probably too much as a standard for theoretical physics. The misunderstanding that mechanics is part of mathematics has produced many beautiful results, think of Euler, Lagrange, Laplace and Riemann, it also obstructed the emergence of new physical ideas. In Poincaré's lectures of 1887–88 he reflects on a large number of optical theories while noting that it does not matter whether the ether exists or not. The ether is just a convenient tool as electromagnetic waves act as if the ether exists. This seems also to have been the point of view of the most important physicist of that time, H.A. Lorentz. Poincaré mentions the ether but also states that at some time the ether will be discarded as useless.

Poincaré lectured in 1908 at the Institute of Post and Telegraph where he taught about the transmission of light and



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wireless telegraph signals. His analysis of self-excited oscillations led to many applications in this engineering field.

One of the conclusions of the author is that Poincaré pushed theoretical physics from the study of particular models to a nuanced understanding of the organizing principles. His interest went further to the development of special relativity and Lorentz transformations in a publication shortly before Einsteins 1905 publication on relativity, see also [5].

Henri Poincaré

The three-body problem

In a second chapter (pp. 51–149) Alain Chenciner describes Poincaré's work on the 'three-body problem'. The basis for this chapter are the *Méthodes nouvelles de la mécanique célèste* and a three-year seminar on reading these books. It correctly emphasizes that the Mécanique Célèste of Poincaré is the first text on the modern theory of dynamical systems with main but not exclusive field of application the three-body problem. After describing the problem formulations and results of Lagrange and Laplace on the stability of the solar system, the author discusses various coordinate systems (a recurring and rather nasty aspect of celestial mechanics) to formulate the basic problem of perturbing simple solutions as periodic ones and tori that have been obtained by simplifying or truncating the complete equations of motion. For centuries, the main tool to handle such questions have been by series expansions, a topic discussed extensively by Poincaré with the two-fold motivation that series expansion should add precision and at the same time should prove existence of the objects described; both aspects are by no means trivial as most expansions in celestial mechanics are formal. Poincaré shows that most expansions are divergent, also the Lindstedt series which he prefers. Lindstedt introduced his series as a formal procedure to approximate a perturbed harmonic equation and a system of two coupled harmonic equations. Poincaré used many chapters to discuss series and their properties, a discussion to which the author of this chapter adds pictures of the type Poincaré could have had in mind. The inexperienced reader might loose his way in the various cases of fixed or varying frequencies, fixed or varying initial conditions, the role of singularities, degeneracies and resonances, but for people working in this field it is a beautiful summary. Integrability questions and integral invariants showing the relation with results by Lie and later Elie Cartan, are placed in the framework of modern theories. Other aspects are the use of normal forms and bifurcations to study periodic solutions and tori, clearly also anticipating singularity theory. It will have been quite an effort to compile this chapter, but notwithstanding the wealth of information we still have to add a few critical remarks.

First, the author does not seem to be aware of the modern theory of averaging and normal forms, see [4] with its results on accuracy and timescales; these basic aspects could have clarified a large number of calculational results. Then, somewhat more emphasis could have been given to the fact that Poincaré–Lindstedt series are convergent for periodic solutions, also for KAM-tori with fixed Diophantine frequencies. As a third item: Birkhoff normal forms were in modern times replaced by Birkhoff–Gustavson normal forms with many new results. Finally, on page 97 the method of Ziglin and other mathematicians is mentioned as the main tool to study non-integrability; this method has been very fruitful qualitatively but it misses out on evidence regarding the dynamics and the nature of non-integrability in which horseshoe maps often play a part. One may find for instance algebraically that a Hamiltonian system is non-integrable but one remains ignorant of the fact that this happens in thin subsets so that the non-integrable effects are negligible or find for instance that non-integrability has significant effects but is very much localised to certain subsets. Other methods are in this respect important and helpful like those of Duistermaat and of Shilnikov-Devaney (for references see again [4] or [5]).

Poincaré's odds

Chapter 3 by Laurent Mazliak (pp. 151–192) describes probability. This field was not one of the main topics of Poincaré's research, but he lectured and published about probability influencing in this way its development, for instance by Borel and Bachelier. Until the end of the nineteenth century, probability got little attention in France and it is typical that, although Poincaré formulated ideas about randomness, he still kept a position between the deterministic views of that time and the allowance of random phenomena. This becomes clear in his approach to thermodynamics and statistical mechanics, emerging fields in his time. Another problem in appreciating his contributions is that Poincaré considered probability problems and made calculations without stating assumptions and producing a general framework.

An early probabilistic statement can be found in [2] (the final text of his famous Prize Essay) where he formulates the Recurrence Theorem. Later, when he turns his attention to thermodynamics this recurrence yields basic problems in the description of systems of many particles. The Maxwell postulate of equipartition of kinetic energy of particles conflicts with recurrence. Is equipartition a transitory state lasting for an extremely long time?

And what about mixing of fluids? A drop of ink into a glass of water will mix, we do not expect that the drop of ink materialises again after some time. Without actually solving all these questions but certainly inspired by them, Poincaré started lecturing on probability. He considered the repetition of experiments that for a large number of times lead to a kind of asymptotic equilibrium. In this resulting distribution, the final state, the initial distribution plays no part. Poincaré applies this to playing roulette, the distribution of asteroids in the solar system and to card shuffling. His calculations are very explicit without reference to theorems known at that time; this is of course typical for Poincaré's style of working and writing.

An interesting discussion concerns the appreciation of the heritage of Poincaré's ideas on probability. His immediate successor in France in this field was Émile Borel who was a star of analysis around 1900. Unexpectedly, Borel started to work on probability, until that time a disreputable topic among mathematicians. He introduced quantification of randomness by using the Lebesgue integral and measure theory. In line with Poincaré's conventionalism he identified probability with the measure of a subset of a suitable set. Much later, the theory of Markov chains became known with contributions by the Czech Hostinsky who was inspired by Poincaré's treatment of the card shuffling problem and by discussions on statistical physics. The author of this chapter summarizes Poincaré's influence on probability as follows:

> He began to extract the domain from the grey zone to which it had been confined by almost all French mathematicians, he initiated methods that flourished when they integrated more powerful mathematical theories, he convinced Borel of the importance of certain questions, to the study of which he eventually devoted an enormous amount of energy. For a rather marginal subject in Poincaré's works, such a contribution appears far from negligible.

Uniformisation of Riemann surfaces

Chapter 4 by François Béguin (pp. 193–230) is for a large part based on the book [3]. The author notes wisely that to understand Uniformisation theory one should ignore the modern formulations and consider the original discussions first. In the nineteenth century, uniformisation was a matter of parametrisation of Riemann surfaces for multi-valued complex functions. (I remember that as a student of complex functions I was baffled by the 'solution' for multi-valuedness by joining Riemann sheets at the problematic positions; how could this be interpreted geometrically, and in what space?) The nineteenth century idea to describe Riemann surfaces was to use parametrisation of curves and surfaces. Poincaré was the first to prove that it is always possible to reduce any algebraic relation, say $F(x,y)=0$, by expressing it as single-valued automorphic functions of one variable. The geometric interpretation of this parametrisation came much later. It is typical that Poincaré's interest in these questions (he was still a lecturer in Caen) came from learning about the treatment of Fuchs of linear ordinary differential equations. The coefficients of the equations have poles so that the complex solutions are naturally multi-valued. A little bit later, after 1880, Poincaré succeeded in proving a more general uniformisation theorem valid for all algebraic Riemann surfaces. It took a long time, until 1907, for Poincaré and Koebe to prove that all Riemann surfaces, algebraic or not, can be uniformised. The author of this chapter traces in detail the discussions and results of the various uniformisation theorems, also the part played in the initial phase by Felix Klein.

Many aspects of the four papers in this monograph can be found in the biographies [1] and [5]. However, the detailed discussions in all of them add a lot of material and understanding to these biographies; for me the third chapter on probability presented a clear and new view on Poincaré's interest in this field. Chapter 2 gives a relatively rare discussion on the problems of studying tori bifurcations by series expansions. The monograph can be a serious help to scientists aspiring to study Poincaré.

References

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3 H.-P. de Saint-Gervais, Uniformisation des surfaces de Riemann, retour sur un théorème centenaire, *École Normale Supérieure Éditions*, Lyon, 2010.

4 J.A. Sanders, F. Verhulst and J. Murdock, *Averaging methods in nonlinear dynamical systems*, Applied Mathematical Sciences, Vol. 59, Springer, 2007, 2nd ed.,

5 Ferdinand Verhulst, *Henri Poincaré, impatient genius*, Springer, 2012.