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A glimpse into the asymptotics of polynomial identities

Many daily phenomena can be modelled by algebraic structures such as algebras. But what do the 'rough shapes' of these abstract objects look like? Commutativity, associativity, nilpotency,... all these important properties can be expressed in polynomials with non-commutative variables. At the 2016 BeNeLux Mathematical Congress the KWG Prize was awarded to Geoffrey Janssens. In this article he writes about his research on algebras satisfying polynomial identities and how they may be distinguished using asymptotic theory.

There is an abundance of examples of functions that arise in our everyday lives and in nature such as cooking and tasting food, washing and drying clothes, going to left or right while driving, stock exchange, et cetera. Note that in the first three examples the order in which the actions take place matters. In other words these functions do not commute. Algebras can be used to model their behaviour. Another, more advanced, example is momentum and position of subatomic particles in quantum mechanics. By the fundamental equation of quantum mechanics they are related by $PM - MP = i\hbar$ where $\hbar$ is Planck's constant. In this case the algebraic model corresponding to this is the so called Weyl Algebra which is generated by two variables x and y and satisfying $xy - yx = 1$.

As a starter we will explain more precisely what is an algebra. After that, we shall address such questions as "What is a polynomial identity?", "What do they tell us?" and "Can we classify all algebras corresponding to a given set of polynomial identities?"

What is an algebra?

A set A is called an algebra over $\mathbb{C}$ if it is a vector space over $\mathbb{C}$ (i.e., one can add elements in A and do scalar multiplication with scalers from $\mathbb{C}$), it is a (associative) ring with unit element (i.e., we can not only add but also multiply two elements from A in a compatible way which is expressed by distribution) and finally $\alpha ab = a\alpha b = ab\alpha$ for all $a, b \in A$ and $\alpha \in \mathbb{C}$.[1] The latter property simply express that also the both types of multiplication are compatible with each other. The easiest example is $\mathbb{C}^n$, the n-dimensional complex space. A more enlightening example is the set consisting of the $n \times n$ square matrices $M_n(\mathbb{C})$ with entries in $\mathbb{C}$. Also $\mathbb{C}[x_1, \ldots, x_n]$, the set of polynomials in commutative variables $x_1, \ldots, x_n$, is an example.

All the examples are instances of *finitely generated algebras over* $\mathbb{C}$. This signifies that there exists a finite number of elements $a_1, \ldots, a_n$ such that $A = \mathbb{C}\langle a_1, \ldots, a_n \rangle$ is generated as an algebra over $\mathbb{C}$ by the elements $a_1, \ldots, a_n$ (i.e., by starting with a finite number of elements $a_1, \ldots, a_n$ and by only multiplying, adding and taking scalar multiples one recovers all of A). In case of $\mathbb{C}^n$ these elements are typically the unit vectors $\vec{e}_i$ for $1 \leq i \leq n$, and in the case of $M_n(\mathbb{C})$ an easy example of a basis is provided by the elementary matrices E_{ij} (having 1 on place (i,j) and 0 elsewhere). Note that in these examples even more holds, because it is enough to take only linear combinations (i.e., only addition and scalar multiplication) of the elements $a_1, \ldots, a_n$, called basis elements, to reconstruct all of A. If such elements with this stronger property exist, A is said to be *finite-dimensional over* $\mathbb{C}$.

Now we have some examples, we could go on and try to describe all possible algebras over $\mathbb{C}$ up to isomorphism (say up to 'renaming symbols'). However, this is completely hopeless. The goal will rather be to describe the possible 'rough shapes', such as commutativity, of algebras. To make this more precise some definitions are needed. Recall that an algebra A is commutative if $ab = ba$ for all $a, b \in A$. Or put otherwise if and only if $f(a,b) = 0$ for all $a, b \in A$ where $f(x,y) = [x,y] := xy - yx$. Such a polynomial is called a polynomial identity of A. For instance $\mathbb{C}^n$ and $\mathbb{C}[x_1, \ldots, x_n]$ satisfy this commutativity polynomial identity.

A first step into polynomial identities

Throughout this article, X will denote an infinite (countable) set of variables, say

$X=\{x_i \mid i\in\mathbb{N}\}$. Further $\mathbb{C}\langle X\rangle$ is the set consisting of all non-commutative polynomials in the variables from X. This is an algebra over $\mathbb{C}$ for the usual addition and multiplication of polynomials.[2]

Definition. A non-zero polynomial $f(x_1,\dots,x_n)\in\mathbb{C}\langle X\rangle$, in some non-commutative indeterminates $x_1,\dots,x_n$, is called a *polynomial identity of A* if $f(a_1,\dots,a_n)=0$ for any $(a_1,\dots,a_n)\in A^n$, notation: $f\equiv_A 0$. The set of all polynomial identities of A is denoted

$$\mathrm{id}(A)=\{f\in\mathbb{C}\langle X\rangle \mid f\equiv_A 0\}.$$

Such polynomials need not exist in general (as is the case for the Weyl algebra[3]). In many cases however, it does, and then A is called a *PI algebra*. To start, consider $A=M_2(\mathbb{C})$, then $f(x,y,z)=[[x,y]^2,z]\equiv_A 0$. To see this we need the Cayley–Hamilton theorem, which asserts that a matrix satisfies its own characteristic equation. For a 2×2-matrix C this equation has the form $x^2-\mathrm{Tr}(C)x+\det(C)=0$ where $\mathrm{Tr}(-)$ denotes the trace of a matrix. Since $\mathrm{Tr}([C_1,C_2])=0$ for two matrices $C_1,C_2\in M_2(\mathbb{C})$ and because $\det(A)I_2$ is a scalar matrix (in particular it commutes with all the other matrices), we indeed get that $[[x,y]^2,z]$ is a polynomial identity of $M_2(\mathbb{C})$.

More generally, $A=M_n(\mathbb{C})$ satisfies some polynomial identity, e.g.

$$f_{n^2+1}:=\sum_{\sigma\in\mathrm{Sym}_{n^2+1}}\mathrm{sgn}(\sigma)x_{\sigma(1)}\cdots x_{\sigma(n^2+1)},$$

called the standard polynomial.[4] As an important consequence *any finite-dimensional algebra satisfies a polynomial identity*. Indeed the regular representation,

$$\rho: A\to GL(A): a\mapsto(\rho_a: A\to A: b\mapsto a\cdot b),$$

embeds any finite-dimensional algebra into matrices of size $\dim(A)$. All this is to say that the class of PI-algebras is a large one.

Enriched structure of $\mathrm{id}(A)$

So PI theory is a story about algebras and their corresponding set of polynomial identities $\mathrm{id}(A)$. This set is actually a (two-sided) ideal of $\mathbb{C}\langle X\rangle$ (i.e., if $f,g\in\mathrm{id}(A)$, then also $f+g$, $h\cdot f$ and $f\cdot h$ are in $\mathrm{id}(A)$ for any $h\in\mathrm{id}(A)$). Further it possess one more important property, namely if $f(x_1,\dots,x_n)\in\mathrm{id}(A)$ and $g_1,\dots,g_n$ are arbitrary polynomials in $\mathbb{C}\langle X\rangle$, then also $f(g_1,\dots,g_n)\in\mathrm{id}(A)$. So polynomial identities remain polynomial identities after eventual substitutions. In more sophisticated words, $\mathrm{id}(A)$ is closed under endomorphisms $\phi\in\mathrm{End}_{\mathbb{C}}(\mathbb{C}\langle X\rangle)$. An ideal with this property is called a *T-ideal*. Now it is not hard to check that all T-ideals of $\mathbb{C}\langle X\rangle$ are actually of this type. In fact, if I is a T-ideal, it is easily proved that $\mathrm{id}(\mathbb{C}\langle X\rangle/I)=I$.[5]

The algebra $\mathbb{C}\langle X\rangle/\mathrm{id}(A)$ is called *a relatively free algebra*.[6] Our story is one about $\mathrm{id}(A)$, but now we see that equivalently, since there is a 1-1 correspondence, it is a story about understanding the different possible relatively free algebras $\mathbb{C}\langle X\rangle/I$ with I a T-ideal in $\mathbb{C}\langle X\rangle$.

Specht's problem and representability

Let A be a finitely generated algebra over $\mathbb{C}$.

Specht's problem. Do there exists polynomials $f_1,\dots,f_l\in\mathrm{id}(A)$ such that $\mathrm{id}(A)=(f_1,\dots,f_l)_{T\text{-id}}$ is finitely generated as a T-ideal?[a]

This is a variant of Hilbert's basis theorem which gives an affirmative answer for the two-sided ideals of the commutative polynomial ring.

It is important to add 'as a T-ideal', since otherwise the result do not hold, e.g. $I=(yx^ny \mid n\in\mathbb{N})$ is a non-finitely generated ideal of $\mathbb{C}\langle x,y\rangle$.

In his seminal work from 1991, Kemer proved Specht's problem [15]. Actually he proved a stronger statement, called the representability theorem.

Representability theorem. *Let A be a finitely generated algebra over a field F. Then there exists a finite-dimensional algebra B that is PI-equivalent to A. Moreover, $F\langle y_1,\dots,y_m\rangle/\mathrm{id}(A)$ can be embedded in a matrix algebra $M_n(L)$ where L is a field extension of F.*

Thus it is not possible to distinguish finitely generated algebras from finite dimensional ones solely using polynomial identities. Also this will enable to assume A is finite-dimensional.

[a] In the T-ideal also all substitutions into the f_i are added.

The classification problem

We now have all the ingredients to refine our thoughts into the following problem:

> "Classify all finitely generated algebras over $\mathbb{C}$ up to PI equivalence"

where we say that A is PI-equivalent to B if $\mathrm{id}(A)=\mathrm{id}(B)$.[7] By the explanations in the previous section we could equally ask to classify up to isomorphism all relatively free algebras.

So, in the above question, we do not take into account 'small relations' only relating certain elements of A. In a way we try to describe the possible rough shapes delivered by polynomial identities to an algebra.

Let us reformulate the problem. First fix some set $S=\{f_i\in\mathbb{C}\langle X\rangle\}$ of polynomials in non-commutative variables from the set X. Associated to it we can consider the set

$$\mathcal{V}(S)=\{A\in\mathrm{Alg}_{\mathbb{C}} \mid S\subseteq\mathrm{id}(A)\},$$

called *the variety*[8] *corresponding to S,* consisting of all algebras (over $\mathbb{C}$) having at least all $f_i\in S$ as polynomial identities. For example if $S=\{xy-yx\}$ then $\mathcal{V}(S)$ is the set of all commutative algebras. Now, in other words, *the goal is to describe $\mathcal{V}(S)$ for all possible sets S.*

How does one tackle such a problem? One way is to find a full list of invariants distinguishing (and thus determining) all varieties. Unfortunately, this is (yet) completely out of reach. An invariant is a number that one associates to any algebra (and variety) and which does not change under PI-equivalence.[9] For example, the area of a triangle is an invariant with respect to isometries of the Euclidean plane. Also the determinant of a matrix associated to a certain linear map $\phi: V\to V$ is invariant under change of basis of V. Moreover this invariant fits perfectly in our mindset. More precisely, the determinant notices the difference between invertible linear maps ($\det\neq 0$) and non-invertible linear maps ($\det=0$).

In our setting, with any algebra A (and variety) we associate a function $c(A):\mathbb{N}\to\mathbb{N}$ which will turn out to look asymptotically (i.e., for n big enough) as the function $f(n)=qn^td^n$. The numbers t and d will be the invariants one is looking for. Moreover the numbers t and d will be (half)-integers and be connected (in a precise way) to the algebraic structure of A. Let us be more concrete.

Asymptotics behind polynomial identities

In order to describe an algebra up to PI-equivalence we have to completely determine $\mathrm{id}(A)$. Unfortunately, determining when an arbitrary polynomial is in $\mathrm{id}(A)$ can be very hard and painful. Luckily, in characteristic 0, it is enough to determine which multilinear polynomials are polynomial identities. A polynomial is called multilinear if the power of each variable occurring is exactly one in each monomial. More formally,

$$P_n(\mathbb{C}) = \mathrm{span}_{\mathbb{C}}\{X_{\sigma(1)}\cdots X_{\sigma(n)} \mid \sigma \in \mathrm{Sym}_n\}$$

is the set of all multilinear polynomials over $\mathbb{C}$. Now let $f \in \mathrm{id}(A)$. Then it is possible, by a multilinearization process, to replace f by a set of multilinear polynomials g_i that are polynomial identities if and only if f is and such that f is in the T-ideal generated by the g_i (e.g. if $f(x) = x^2$, then $g(x,y) = f(x+y) - f(x) - f(y) = 2xy$ and $f(x) = \frac{1}{2}g(x,x)$). Thus if we know $\mathrm{id}(A) \cap P_n(\mathbb{C})$ for all n, then we can also reconstruct whole of $\mathrm{id}(A)$ and consequently the PI-equivalence class of A. Without real surprise, the story would have been too short otherwise, for only in very few cases generators for $\mathrm{id}(A)$ and $\mathrm{id}(A) \cap P_n(\mathbb{C})$ are known. Even for $M_n(\mathbb{C})$ with $n \geq 3$ this is an open problem. Instead one could try to compute only $\dim_{\mathbb{C}} \mathrm{id}(A) \cap P_n(\mathbb{C})$ for large n, which a priori is more tractable (but provides less information). Note that $\dim_{\mathbb{C}} P_n(\mathbb{C}) = |\mathrm{Sym}_n| = n!$. It turns out that for n big enough also $\dim_{\mathbb{C}} \mathrm{id}(A) \cap P_n(\mathbb{C}) \approx n!$ which is asymptotically a wild function.

In the light of all this, much research in the field of asymptotic PI-theory is focused on the sequence $(c_n(A))_n$, where

$$c_n(A) = \dim_{\mathbb{C}} \frac{P_n(\mathbb{C})}{P_n(\mathbb{C}) \cap \mathrm{id}(A)},$$

is called the *n-th codimension* of the algebra A. Notice that the function $c(A) : \mathbb{N} \to \mathbb{N} : n \mapsto c_n(A)$ depends on $\mathrm{id}(A)$ rather then A, thus it is constant on PI-equivalence classes and can therefore be used as an invariant. Unfortunately determining $c_n(A)$ in any point is, even for concrete examples, out of reach. Nevertheless for large n it becomes tractable. The reason for this is the pioneering result of Regev who proved in 1972 that this function is exponentially bounded [16], i.e.,

$$\exists d \in \mathbb{R}: \quad c_n(A) \leq d^n \text{ for all } n.$$

Historically Regev introduced codimensions and used above exponential bound in order to solve in the affirmative sense an (at that time) sixty years old conjecture asserting that the tensor product of two PI algebras is again PI.

At this stage of the story we can associate to any PI algebra a sequence of numbers $c_n(A)$ such that $\limsup_{n\to\infty} \sqrt[n]{c_n(A)}$ exists. But much more has to come.

Conjecture of Regev and Amitsur

Concerning the exact asymptotics of the codimension sequence of a finitely generated algebra, Regev conjectured at the end of the seventies the following.

Conjecture (Regev, 70's). *There exist numbers $t \in \frac{1}{2}\mathbb{Z}, d \in \mathbb{N}$ and $c \in \mathbb{Q}[\sqrt{2\pi}, \sqrt{b}]$ for some $b \in \mathbb{N}$ such that*

$$c_n(A) \simeq cn^t d^n$$

where $f \simeq g$ iff $\lim_{n\to\infty} \frac{f}{g} = 1$.

Historically one should mention an even earlier conjecture of Amitsur asserting that the limit $d = \lim_{n\to\infty} \sqrt[n]{c_n(A)}$ exists and is an integer. This number, which represents the exponential growth rate of $c(A) : \mathbb{N} \to \mathbb{N} : n \mapsto c_n(A)$, is called the *PI-exponent of A*, denoted $\exp(A)$.

This integrality conjecture is very surprising if one thinks of other growth functions in algebra. As an illustration, one aspect of this conjecture is that the codimension sequence would (asymptotically) never be a function between a polynomial and an exponential function such as $f(n) = e^{\sqrt{n}}$. This is in contrast to other growth functions such as the word growth in group theory. Also the polynomial growth of the word growth function of a finitely generated algebra (which gives rise to the so called Gelfand–Kirillov dimension) can be any real value greater than 3. Thus the conjectures of Amitsur and Regev are really strong ones.

It was only in 1998, in their breakthrough paper [6], that Giambruno and Zaicev proved that indeed

$$d = \lim_{n\to\infty} \sqrt[n]{c_n(A)} \in \mathbb{N}.$$

This is an amazing fact, but nevertheless one could wonder whether this number contains any useful information... The answer is yes! They proved the integrality by delivering a surprisingly transparent and computable algebraic formula relating d to the Wedderburn–Malcev decomposition of A (in case A is finite-dimensional). Philosophically, d relates how the 'nice elements' (i.e., semisimple) and the 'bad elements' (i.e., the Jacobson radical $J(A)$) in A interact with each other.

It is time for two small examples. First suppose A is abelian and thus $xy - yx \in \mathrm{id}(A)$. In this case $d = 1$. To prove this we must find a basis of $\frac{P_n(\mathbb{C})}{P_n(\mathbb{C}) \cap \mathrm{id}(A)}$ as vector space over $\mathbb{C}$. In this quotient space, one has $x_i x_j - x_j x_i = 0$ (and thus $x_i x_j = x_j x_i$) for any variables x_i and x_j. Consequently, in this case $\frac{P_n(\mathbb{C})}{P_n(\mathbb{C}) \cap \mathrm{id}(A)} = \mathrm{span}_{\mathbb{C}}\{x_1 \cdots x_n\}$.

Wedderburn–Malcev decomposition

Representation theory aims to represent groups and algebras inside matrix algebras $M_n(\mathbb{C})$ which we understand well from our first courses in linear algebra. These are, in a ring theoretic sense, simple and direct summands $M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_l}(\mathbb{C})$ are called semisimple algebras.[a] Unfortunately in general an algebra is not semisimple. One can collect all 'bad elements' due to which A fails to be semisimple. This set, which actually is an ideal, is called the Jacobson radical, denoted $J(A)$.[b]

Theorem (Wedderburn–Malcev). *Let A be a finite-dimensional algebra over $\mathbb{C}$. Then*

$$A = B_1 \oplus \cdots \oplus B_l \oplus J(A)$$

where B_i is a simple subalgebra (say $B_i \cong M_{n_i}(\mathbb{C})$), $B_1 \oplus \cdots \oplus B_l$ a maximal semisimple subalgebra, $J(A)$ is nilpotent (i.e., there exists a number $s \in \mathbb{N}$ such that $J(A)^s = 0$) and $\oplus$ the direct sum of vector spaces.

Thus one can properly decompose the elements into a set of 'nice elements' (the semisimple part) and of 'bad elements' (the radical part).

[a] To be more precise, an algebra A is semisimple if and only if it is a direct sum of minimal left ideals. By a theorem of Wedderburn–Artin semisimple $\mathbb{C}$-algebras are isomorphic to a direct sum (of rings) $M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_l}(\mathbb{C})$.

[b] Concretely $J(A)$ is the intersection of all (left) maximal ideals. Moreover, $J(A)$ is the smallest ideal I such that A/I is semisimple.

Therefore $c_n(A)=1$ for any n and indeed $d=\lim_{n\to\infty}\sqrt[n]{c_n(A)}=1$. In the case of our other main example $M_m(\mathbb{C})$, the exponential growth rate $d=m^2$ equals the dimension of the algebra.[10]

Next, using topological methods, Berele and Regev proved in 2008 the full conjecture except for the part $c\in\mathbb{Q}[\sqrt{2\pi},\sqrt{b}]$, see [4]. Unfortunately this time no concrete information concerning t can be extracted from the proof. Therefore it remained as a main open problem in asymptotic PI theory to understand this black box. Finally, in October 2015, joint with Yakov Karasik and Eli Aljadeff, we found a concrete formula for the polynomial growth rate t [2]

Time has come to go in detail on the concrete information contained in the invariants t and d.

The first invariant d

Let A be a finitely generated algebra over $\mathbb{C}$. By the representability theorem we may even assume A to be finite-dimensional. In particular we can decompose it nicely according to the theorem of Wedderburn–Malcev $A=B_1\oplus\cdots\oplus B_q\oplus J(A)$. Then

Theorem (Giambruno–Zaicev [6]). *With notations as before,*

$$d=\max\dim\{B_{i_1}\oplus\cdots\oplus B_{i_r}\mid B_{i_1}J(A)\cdot\cdots\cdot J(A)B_{i_r}\neq 0\}$$

where $r\geq 1$ and all B_{i_j} are different simple components.

So, as announced, the number d is tightly connected to the algebraic structure and the way 'bad' and 'good' elements interact with each other. With this formula at hand it is clear that, as mentioned earlier, if $A=M_m(\mathbb{C})$ then $d=m^2$ (since then $J(A)=0$). More generally, if A is the algebra $UT(d_1,\dots,d_q)$ consisting of upper block triangular matrices of the type

$$\begin{pmatrix} M_{d_1}(\mathbb{C}) & & * \\ 0 & \ddots & \\ \vdots & & \\ 0 & \cdots\ 0 & M_{d_q}(\mathbb{C}) \end{pmatrix},$$

then $d=d_1^2+\cdots+d_q^2$. Finally, if A is a so-called quiver algebra, then d tells us what is the largest path in the quiver that does not pass the same vertex twice.

Giambruno and Zaicev, in [7], also handled the case when A is not a finitely generated algebra (by using a generalization of the representability theorem).

The second invariant t

As before we may decompose $A=B_1\oplus\cdots\oplus B_q\oplus J(A)$ according to Wedderburn–Malcev's theorem. Then:

Theorem (Aljadeff, Janssens and Karasik [2]). *If A is a finite-dimensional basic algebra (see definition below), then*

$$t(A)=\frac{d-q}{2}+(s-1)$$

where $d=\sum_{i=1}^{q}\dim B_i$ and $s\in\mathbb{Z}_+$ the smallest integer such that $J(A)^s=0$.

The proof uses, among other things, Kemer Theory (i.e., techniques and objects central in the solution by Kemer of Specht's problem and his representability theorem) and introduced the so-called basic algebras, which can serve as building blocks for decomposing algebras up to PI-equivalence.

Basic algebras

With any finite-dimensional algebra, due to the theorem of Wedderburn–Malcev, we can associate two numbers d and s where d and s are as in the theorem above. The tuple $\mathrm{Par}(A)=(d,s)$ is called the *parameter* of A. Basic algebras are minimal models for a certain given tuple (d,s) of numbers $d,s\in\mathbb{N}$. More precisely:

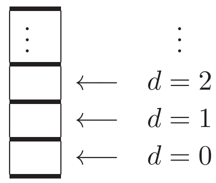
Definition. A finite-dimensional algebra A is called *basic* if A is not PI equivalent to an algebra $B=B_1\times\cdots\times B_r$ where B_i are finite-dimensional algebras such that $\mathrm{Par}(B_i)<\mathrm{Par}(A)$ for any $i=1,\dots,r$.

These algebras have the advantage to yield geometric and combinatorial translations. I will not go further in detail and rather refer to [3]. Prime examples of basic algebras are matrix algebras $M_m(\mathbb{C})$ and upper block triangular matrices $UT(d_1,\dots,d_l)$. Also, any (finite-dimensional) algebra is PI-equivalent to a direct product of basic subalgebras. Since the polynomial growth rate t behaves well towards direct products above result gives an interpretation to t for any finitely generated algebra.

Classifying varieties

In the next stage of the story, now that we have these invariants containing useful algebraic information on A, it is time to use them for the problem of classifying varieties (cf. section 'The Classification problem').

For this we start by distributing all varieties into layers according to their PI-exponent (see the figure).



Let now S be a fixed set of polynomials, consider $\mathcal{V}(S)$ and suppose that its exponential growth is d.[11]

It could be that by adding polynomials to S we get a strictly smaller variety (since not all algebras in $\mathcal{V}(S)$ have to satisfy this extra polynomial) with a strictly smaller invariant d. If this always happens, $\mathcal{V}(S)$ is called a *minimal variety* (intuitively in this case $\mathcal{V}(S)$ lies at 'the bottom of the layer'). They have been classified in [8] and the answer surprisingly turns out to be very elegant. Namely:

Theorem (Giambruno–Zaicev). *Let $\mathcal{V}$ be a variety with $\exp(\mathcal{V})\geq 2$.[12] Then $\mathcal{V}$ is a minimal variety if and only if there exists a upper block triangular matrix algebra $UT(d_1,\dots,d_q)\in\mathcal{V}$ such that $\mathcal{V}=\mathcal{V}(\mathrm{id}(UT(d_1,\dots,d_q))$.*

In a next step one tries to differentiate varieties in a fixed layer (i.e., we fix the invariant d). Again we can distribute them into smaller layers depending on the polynomial growth, thus the invariant t. Investigations are being done into classifying the varieties minimal with respect to the invariant t with a fixed exponential growth d. There is evidence that again an elegant answer pops up.

Other and further research

At this point we have a beautiful story starting with a set of algebras sharing the same 'rough shape' (delivered by a common set of polynomial identities) to which we can associate two invariants (which basically are the exponential and polynomial growth rate of the so called codimension sequence) that delivers precise information. Due to these we are able to differentiate certain classes of algebras. But what next?

Two remarks can be made about the story. To begin with, by the representability theorem it is sufficient to consider finite-dimensional algebras. The proof of this theorem, however, is not constructive. Thus an important, and completely open, problem is to find an algorithm whose input is a finitely generated algebra and the output some finite-dimensional one PI-equivalent to it.

In the same spirit, for the polynomial part, we use a decomposition into basic algebras. Again it would be interesting to have a more constructive proof of this. Actually more examples of basic algebras would already be welcome.

The ultimate goal remains to classify all algebras up to PI-equivalence. In full generality this problem seems to be completely out of reach. However, as Steve Jobs said: "Stay hungry stay foolish!" Towards a fuller answer researchers have (successfully) started to take more refined information, such as certain (algebraic) group or Hopf algebra actions or (semi)group gradings, into account. Some of the most recent results can be found in [1, 10, 11, 14] and the references therein. First things first, a good starting reference to learn asymptotic PI-theory is [9].

Going beyond the above associative setting, we note that all definitions make sense for non-associative algebras, such as Lie algebras. For a survey of the progress on the analoguous conjectures in this setting we refer to [5, 12, 13, 17] and the references therein. Last but not least, to find methods for finding generators of $\mathrm{id}(A)$ is one of the main open challenges.

As a brief summary, in spite of many nice results and joyful partial answers concerning the classification problem, a long and interesting road towards describing all varieties (and thus all algebras up to PI-equivalence) is ahead of us. But there is nothing as nice as a walk on a sunny day!

Notes

1 Actually everything holds over an arbitrary field of characteristic 0. But for sake of clarity we simply consider $\mathbb{C}$.

2 In other words, $\mathbb{C}\langle X\rangle$ is the free algebra over $\mathbb{C}$ generated by the elements $x \in X$.

3 If it would satisfy a polynomial identity then the Weyl algebra would have finite dimensional simple representations. However, the Weyl algebra has no finite dimensional representations. The latter can be seen by taking the trace of the equation $xy - yx = 1$.

4 This roughly follows from the following two observations: on the one hand, f_{n^2+1} is multilinear, so it is sufficient to substitute the basis elements of $M_n(\mathbb{C})$, and there are n^2 such elements. On the other hand, the polynomial is alternating, so if we substitute two times the same element the polynomial vanish.

5 By $\mathbb{C}\langle X\rangle / I$ is meant the quotient $\mathbb{C}\langle X\rangle$ by the ideal I. Intuitively this means that all elements in I are made equal to zero in $\mathbb{C}\langle X\rangle$.

6 The name is not a coincidence. Actually $\mathbb{C}\langle X\rangle / \mathrm{id}(A)$ is the free object in the category consisting of the algebras B with $\mathrm{id}(A) \subseteq \mathrm{id}(B)$.

7 PI-equivalence is really an equivalence relation.

8 This is a variety in the sense of Birkhoff. Also note that these varieties encompass the equivalence classes of the PI-equivalence relation. Indeed $A \sim_{PI} B$ iff $\mathcal{V}(\mathrm{id}(A)) = \mathcal{V}(\mathrm{id}(B))$ where $\mathcal{V}(\mathrm{id}(A)) = \{C \in \mathrm{Alg}_{\mathbb{C}} \mid \mathrm{id}(A) \subseteq \mathrm{id}(C)\}$.

9 Thus some number that is constant on each equivalence class.

10 Even more holds. Recall that an algebra A is simple if and only it the only two-sided ideals are $\{0\}$ and A. By a theorem of Wedderburn such $\mathbb{C}$-algebras are isomorphic to some $M_n(\mathbb{C})$. It is known that A is simple if and only if $\exp(A) = \dim(A)$. Thus the PI-exponent detects simple algebras out of a set of algebras.

11 Note that by the affirmative answer on Specht's problem we may assume S to be a finite set. If $\mathcal{V}$ is some variety then $\mathrm{id}(\mathcal{V})$ is defined as $\bigcap_{A \in \mathcal{V}} \mathrm{id}(A)$. Thus in our case $\mathrm{id}(\mathcal{V}(S)) = (S)_{T\text{-}\mathrm{id}}$ is the T-ideal generated by S. Using this definition it makes sense to look at $c_n(\mathcal{V}) = \dim_{\mathbb{C}} \frac{P_n(\mathbb{C})}{P_n(\mathbb{C}) \cap \mathrm{id}(\mathcal{V})}$ and its exponential and polynomial growth rates.

12 Note that $\exp(\mathcal{V}) = 1$ is the same as saying that $\mathcal{V}$ has polynomial growth.

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