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**Research** Vidi project

# Phase transitions, Euclidean fields and self-similar random fractals

**In his Vidi project 'Phase transitions, Euclidean fields and self-similar random fractals,' Federico Camia worked on the mathematical theory of phase transitions in two dimensions, focusing in particular on its connections to Euclidean field theory and on the emergence of geometric structures that are both random and fractal. This article will not attempt to reproduce or describe in detail the results of his research; it will rather take inspiration from that research to explore how combining self-similarity and randomness produces interesting objects whose analysis requires a mixture of techniques from different areas of mathematics.**

Fractal-like structures are common in nature. They have a property called self-similarity, which means that each portion looks like the whole. Some of the most famous mathematical fractals, such as the Cantor, Julia and Mandelbrot sets or the Sierpinski carpet, are produced by a deterministic process and contain identical, scaled-down copies of themselves. On the contrary, natural fractal-like objects usually look random and are self-similar only in a statistical sense, which means that each portion of the object looks similar but not identical to the whole. Besides appearing naturally in the theory of phase transitions, random fractals have many applications to various fields of mathematics, the natural sciences and economics, and they appear in such diverse contexts as the modeling of financial markets and cosmology.

### Critical phenomena and scale invariance

A physical system is said to have *correlation length* $l$ if two portions of the system at distance larger than $l$ are roughly uncorrelated. The correlation length of a system is typically determined by the strength of thermal fluctuations, which are random and tend to 'wash out' the correlation that would normally exist between separate parts of the system. This concept is central in the study of certain phase transitions that lead to the emergence of self-similar patterns.

*Phase transition* is the term used to denote phenomena such as the melting of ice and the freezing or the evaporation of water. In certain types of phase transitions, the correlation length diverges and a sudden 'structural' change takes place as a parameter (e.g., temperature or density) governing the system's behavior goes through a *critical point*. What is meant by the divergence of the correlation length is a special balance between thermal fluctuations and correlations that makes the system effectively scale-free, or scale-invariant. Scale invariance can thus be considered a new symmetry acquired by the system at the critical point. This symmetry implies that the thermodynamic behavior of critical systems must be described by power laws, since other types of laws would not be compatible with scale invariance. (In other words, at the critical point, the thermodynamic functions are powers.) It also implies that certain geometric objects that are sometimes useful to describe some aspects of certain systems, such as clusters and interfaces, must have a fractal nature. Examples of such clusters and interfaces will be provided later, through the introduction of models such as percolation and the Ising model. But, in general, their shapes are determined by random fluctuations, so that, at the critical point, they are both random and fractal-like.

The term *critical phenomena* describes the collection of phenomena, such as power law behavior of thermodynamic functions and fractal behavior of certain geometric objects, that take place at the critical point; they are mainly due to the divergence of the correlation length and

**Figure 1** Romanesco broccoli. Fractal-like structures are abundant in nature.

have some universal features that are independent of the specificities of the system or model under consideration. Because of their universality, critical phenomena are well-suited to be analyzed mathematically, since they can be studied using simplified models.

Examples of fractal-like objects are not difficult to find in nature, as evidenced by Figure 1. In mathematics, the prototypical example of a fractal is the triadic Cantor set, depicted in Figure 2. At each iteration, the middle third of each segment is removed according to a scale-invariant construction that looks the same at all scales. For example, looking at Figure 2, what lies below each of the two equal segments produced by removing the middle third of the original segment is a copy of the whole construction scaled down by a factor $1/3$. The total length removed in the construction of the triadic Cantor set is

$$\frac{1}{3}+\frac{2}{9}+\frac{4}{27}+\cdots+\frac{2^{n-1}}{3^n}+\cdots=1.$$

This tells us that the remaining set has total length zero, but it is by no means empty, since it contains, for example, all points of the type $1/3^n$, and (uncountably) many more. What is then the 'size' of the triadic Cantor set? To answer the question we need to decide what concept of size we want to use. Length is clearly not appropriate, since it gives a trivial answer that does not reflect the fact that the Cantor set is not empty.

Let's compare the triadic Cantor set with a 'full' segment, which is clearly a one-dimensional object. If we want to cover a segment of length $1$ with smaller segments of length, say, $\ell_k = 1/3^k$, we need exactly $N_k = 3^k$ such segments. In the case of the triadic Cantor set instead, we need $N_k = 2^k$ segments of length $\ell_k = 1/3^k$. One way of thinking about the dimension $D$ of an object is via the relation $N_k \ell_k^D = 1$. For the segment this gives the familiar dimension $D = 1$, but for the Cantor set it gives $D = \frac{\log 2}{\log 3} \approx 0.631$.[1]


**Figure 2** The first few steps in the construction of the triadic Cantor set.

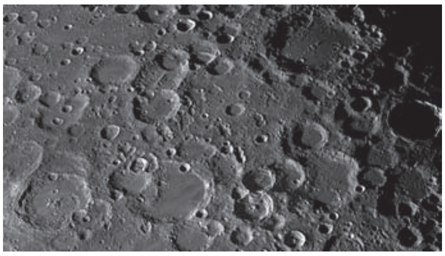

**Figure 3** Benoit B. Mandelbrot noticed that the distribution of the sizes of lunar craters looks approximately scale-invariant [7].

A canonical example of fractality associated with randomness is Brownian motion. Named after the botanist Robert Brown, Brownian motion refers to the jittery motion of particles suspended in a fluid. In 1905, Albert Einstein published a paper explaining how the random motion observed by Brown is a result of the particles colliding with the atoms or molecules in the fluid. Einstein's explanation provided a confirmation of the atomic theory of matter.

The mathematical model of Brownian motion has numerous real-world applications, one of the best-known being in finance, where it is used to model stock market fluctuations (e.g., in the Black–Scholes model). From a mathematical perspective, Brownian motion is a continuous-time stochastic process with continuous trajectories and with increments that are independent and identically distributed. A salient feature of a Brownian trajectory is its scale invariance, which is responsible for its fractal nature. If one takes a portion of a Brownian trajectory and scales it up to the size of the whole trajectory, the result looks the same as the original, but only *in a statistical sense*. Indeed, because of randomness one cannot expect a closer resemblance. The same type of phenomenon can be found on the surface of the moon, as shown in Figure 3.

### The Ising model and its phase transition

In order to introduce random fractal objects more relevant to the Vidi project discussed in this article, we need to turn to statistical mechanics. Historically, statistical mechanics originates from attempts to use the laws of mechanics to derive the laws and equations of thermodynamics. The latter can be seen as either empirical laws (e.g., the ideal gas law: $PV = nRT$) or axiomatic principles that form the basis of the theory (the three laws of thermodynamics). All of them contain only macroscopic variables such as energy, entropy, temperature, volume, and pressure, and make no reference to the microscopic constituents of matter, such as molecules and atoms, and their (quantum-)mechanical interactions. The idea behind statistical mechanics is that one can arrive at the laws of thermodynamics by way of a statistical analysis of the collective behavior of the microscopic constituents of matter. Thus, in statistical mechanics one typically studies systems with a large number of elementary components, and often resorts to a useful mathematical idealization, called the *thermodynamic limit*, in which the number of elementary components is sent to infinity. In doing this, the focus is on the average behavior of the system and on possible deviations from it.

One of the most interesting and challenging topics in statistical mechanics is the study of phase transitions.[2] A *phase* is a state of a thermodynamic system with spatially uniform physical properties, and a *phase transition* is the transformation of a thermodynamic system from one phase to another by heat transfer. A classic example is that of a liquid turning to vapor or vice versa. Another example of a phase transition is what happens to a magnet when warmed to a sufficiently high temperature (depending on the material). If the temperature is high enough, the magnet looses its ability to attract metals; this signals a transition from the *ferromagnetic* to the *paramagnetic* phase.

Let's now explore this phase transition more in detail using a simplified model based on the observation that the atoms inside a magnet act as little magnets themselves, and that it is energetically advantageous for nearby atoms to have their 'north poles' aligned.[3] More precisely, the atoms of a magnet have magnetic moments which naturally tend to align with each other because of magnetic interactions. This tendency is contrasted by random thermal fluctuations which tend to disrupt the 'magnetic order'. The *Ising model* was introduced in 1920 by physicist Wilhelm Lenz, who asked his student, Ernst Ising, to study its behavior at different temperatures. I will describe the two-dimensional version of the model based on a square grid, where the vertices of the grid represent the positions of the atoms (see Figure 4). The three-dimensional version of the model is analogous to the two-dimension-

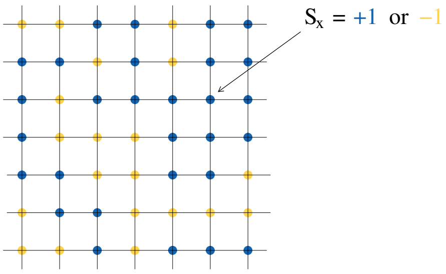


**Figure 4** The two-dimensional Ising model.

al one except that it uses a cubic grid, but much less is known about its behavior. The one-dimensional version, defined on subsets of $\mathbb{Z}$, is the one studied by Ising, but it does not have a phase transition and is therefore uninteresting for our purposes.

To each vertex $x$ of a square grid, one associates a *spin* variable $S_x$ that takes values $+1$ (represented by a darker color, blue, in Figure 4 and 5) or $-1$ (represented by a lighter color, yellow). An assignment of a spin $+1$ or $-1$ to each vertex is called a *spin configuration* and is denoted by $\mathbf{S}$, while $\mathcal{S}$ will denote the set of all spin configurations. On a finite grid, to a spin configuration $\mathbf{S} \in \mathcal{S}$, one assigns probability

$$P(\mathbf{S}) = \frac{1}{Z}\exp\left(\beta \sum_{x,\,y:\,|x-y|=1} S_x S_y\right), \quad (1)$$

where $\beta \geq 0$ represents the inverse of the temperature $T$ ($\beta = 0$ corresponds to the idealization of 'infinite temperature'), the sum is over all pairs of vertices at distance one from each other, and

$$Z = Z(\beta) = \sum_{\mathbf{S} \in \mathcal{S}} \exp\left(\beta \sum_{x,\,y:\,|x-y|=1} S_x S_y\right)$$

is a normalizing factor which ensures that (1) defines a probability distribution.

The exponential form of the probability distribution (1) is very important in statistical mechanics, where it is called a (*Boltzmann-*)*Gibbs distribution*. $Z$ is called the *partition function* of the model and, despite being introduced here simply as a normalization factor, plays a crucial role in statistical mechanics. The partition function depends in general on the (inverse) temperature and possibly on other quantities, depending on the physical system or the mathematical model under consideration. From this dependence one can derive quantities, such as the (average) energy, that describe the thermodynamic properties of the system.

It's not difficult to see that $P$ assigns greater probability to spin configurations where neighboring vertices have the same spin value (or color). Indeed, the two configurations where all spins are $+1$ or $-1$ are always the most probable ones. Informally, one could say that vertices prefer to have the same color, or that a vertex has to pay a 'price', in terms of probability, for 'disagreeing' with its neighbors. But the extent to which disagreement is penalized depends on the value of $\beta = 1/T$. For instance, in the idealized case of infinite temperature, where $\beta = 0$, $P$ is constant and all configurations are equally likely. In such a situation, a 'typical' configuration would look very different from those with all spins $+1$ or all spins $-1$, simply because there are only two such 'orderly' configurations and many more 'mixed' configurations. In this case, what is 'typical' is determined uniquely by chance (or entropy, to be more precise). However, if one increases the value of $\beta$, the 'price' to pay for 'disagreement' increases, giving an advantage to more 'orderly' configurations.

Thus, the model captures, at least qualitatively, the tendency of the magnetic moments of the single atoms to be aligned with each other, as well as the observation that this tendency is less pronounced at high temperatures. But can the model reproduce the transition between the ferromagnetic phase and the paramagnetic phase, marked by the loss of the ability to attract metals as the temperature becomes high enough?

To answer that question we should look at the total magnetization, the sum of all magnetic moments within a finite region. (A more precise definition of the magnetization will be given in the next section, together with a discussion of some results concerning the magnetization and its important role in the analysis of the Ising model.) If thermal fluctuations win over the tendency of neighboring atoms to align their magnetic moments, the latter will average out at large scales, leaving zero total magnetization in any large finite region. The presence of a non-zero magnetization, on the contrary, implies long-range order. It is the presence of a non-zero total magnetization that gives ferromagnets the property of attracting metals.

Ising discovered that in one dimension thermal fluctuations dominate at every non-zero temperature. Hence, the one-dimensional Ising model does not have a phase transition. But the one-dimensional case turns out to be special,[4] and the Ising model in any dimension higher than one exhibits a phase transition. This was first demonstrated by Rudolf Peierls, who showed that in two dimensions the total magnetization in any large region averages to zero at sufficiently high temperatures, and takes a non-zero value at sufficiently low temperatures. The same is true in dimensions three and higher. This is illustrated by Figure 5, which depicts the outcomes of three simulations of the model on a large square grid with a small mesh size, so that the figure can fit the page. (The simulations were carried out by Wouter Kager.)

The simulation on the left corresponds to low temperature and shows the ferromagnetic phase in which the spins tend to be aligned with each other. In this phase, the system 'chooses' one of the two colors with equal probability, and most of the spins take that color. The high-temperature or paramagnetic phase is shown on the right. The contrast is very clear, with the spin configuration on the right looking much more 'chaotic' than that on the left: no order can be seen and there is no preferred color. The reason for this 'chaos' is that entropy wins over energy. If one were to choose a spin configuration uniformly at random (i.e., at infinite temperature, or $\beta = 0$),

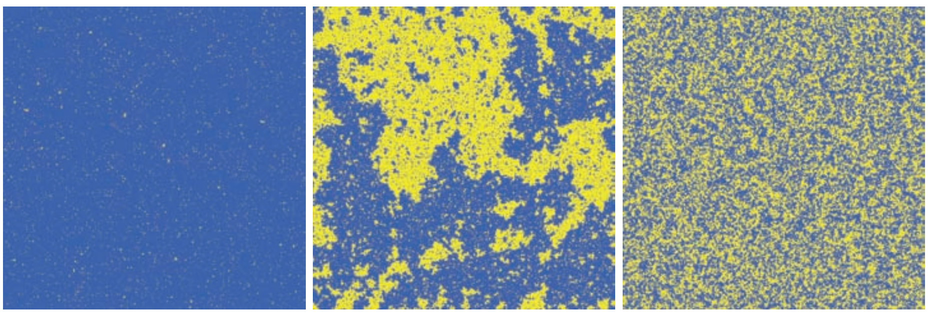

**Figure 5** The Ising model phase transition.

with very high probability the configuration would look like the one on the right rather than that on the left, simply because there are many more configurations that look like that. This is still true at high enough temperatures, when $\beta$ is sufficiently small. But when $\beta$ becomes large, energy considerations become more important and the system tends to 'choose' a more 'ordered' configuration. Intuitively, one can think of energy as a currency, and of disordered configurations as being more expensive than ordered ones. At high temperatures the system has a lot of energy and can afford to be in an 'expensive' configuration, but at low temperatures it is forced to choose a 'cheap' configuration.

We can conclude that the Ising model is successful in reproducing the paramagnetic-ferromagnetic phase transition. This ability has made it one of the most studied models of statistical mechanics. Indeed, the Ising model has played a crucial role in the development of statistical mechanics itself and the theory of phase transitions, and it continues to be a topic of active research.

So far we have focused on the high and low temperature configurations and we haven't said anything about the configuration in the middle of Figure 5, which looks very different from the other two. If one were to start at high temperature and then gradually lower the temperature, one would see the system move from disordered to more ordered configurations. In doing so, the system would have to make a choice of color, breaking the symmetry between blue and yellow, or $+1$ and $-1$.[5] This very important phenomenon is called *spontaneous symmetry breaking* and plays a crucial role not only in the theory of phase transitions but also in particle physics and cosmology.[6]

The configuration in the middle of Figure 5, in which both colors are roughly equally represented, corresponds to the *critical temperature*, when the phase transition happens and the $\pm$ symmetry is about to be broken; it represents the watershed between order and disorder, and has very special properties. In fact, I will argue that it bears some resemblance with Figure 3. In that figure, if you take a small part of the whole and blow it up to the size of the original, what you see looks roughly (statistically) similar to the original. The same is true for the middle configuration in Figure 5 because it contains blue and yellow clusters of all possible sizes between the size of the grid's mesh and that of the whole system, so that a scale transformation will not make a significant difference, as long as the scale factor is small compared to the inverse of the mesh size. The same is not true in the left and right configurations because in both the yellow clusters have a definite mean size, which would be visibly changed by a scale transformation. As a consequence of the 'democracy of scales' of the middle configuration, one can see that the interfaces between blue and yellow clusters look fractal-like, particularly around large clusters, due to their 'peninsulas' and 'fjords'. This is precisely what one would expect from the theory of critical phenomena briefly discussed at the beginning of the introduction. But one cannot have full scale invariance and real fractals on a grid: the presence of the grid will eventually be revealed if one blows up the system by a sufficiently large scale factor.

In order to bring the fractal-like aspects of the critical spin configuration to full-blown scale invariance, one can send the mesh size of the grid to zero. This procedure is called a *scaling limit* and is similar to the thermodynamic limit mentioned earlier in that the number of elementary components of the system (in this case, the number of vertices of the grid) tends to infinity. The difference is that the result of a scaling limit is a *continuum model*, not a model defined on an infinite grid. The advantage of this formulation over the thermodynamic limit is that a continuum model can potentially posses more symmetries than a model based on a grid, such as full scale invariance and rotation invariance. In the 1970s–80s, Polyakov observed [9], based on heuristic physics arguments, that a continuum model obtained by taking the scaling limit at the critical point should not only be invariant under translations and rotations, but under all transformations that leave angles unchanged, technically called *conformal transformations*. This observation provides a tremendously powerful tool to study continuum models, and led to the development of *Conformal Field Theory* (CFT). (I will say more about *conformal fields* in the next section.)

Although simulations and physics arguments give a convincing picture of what to expect from a scaling limit at the critical point, a complete mathematical understanding of scaling limits has proved difficult. The main goal of my Vidi project was to explore the theory of scaling limits at or near the critical point for the two-dimensional Ising model and related models of statistical mechanics, with a new approach that relies on the study of certain random geometric objects such as the clusters and interfaces shown in Figure 5. One particular goal of the research was to provide a rigorous proof of the connection between scaling limits performed at or near the critical point and *Euclidean field theory*, a close relative of the *relativistic quantum field theory* developed by physicists to describe the nature of elementary particles and their interactions.

Despite being one of the most interesting aspects of the theory of scaling limits, and well supported by heuristic physics arguments, until recently there was little mathematically rigorous evidence for this connection beyond some well-understood but relatively trivial examples. One of the main achievements of my Vidi project was to rigorously prove that connection in the case of the two-dimensional Ising model. This is conceptually important because of the crucial role played by the Ising model in the development of statistical mechanics and the theory of phase transitions. In the next and final section, I will describe some of the results about the Ising model obtained during my Vidi project.

**Discussion of some of the main results**

This section is going to be somewhat more technical than the previous ones, but I will try to keep the discussion as simple as possible. One of the quantities analyzed in my Vidi project is the Ising *magnetization*, already mentioned in the previous section. Mathematically, the magnetization inside a domain $D$ is the sum of the spin variables $S_x$ for all $x$ in $D$. Physically, this corresponds to the total magnetic moment generated by all the atoms in $D$. It is an important quantity because its behavior is different in the paramagnetic and ferromagnetic phases, so it can be used to 'detect' the phase transition. In physics jargon it is referred to as the *order parameter* associated with the phase transition. The difference in behavior should be present also after taking the scaling limit described above. But what does it mean to take the scaling limit of the magnetization?

To answer that question, let's consider the Ising model on a large (possibly infinite) square grid. We can think of the vertices of the grid as (a subset of) $a\mathbb{Z}^2$, where $a>0$ is the mesh size of the grid. Consider now a bounded domain $D\subset\mathbb{R}^2$; the *total magnetization* in $D$ is given by

$$M_D^a = \sum_{x\in a\mathbb{Z}^2\cap D} S_x,$$

where the sum is over all vertices contained in $D$, so that all spins contained in that region, and only those, contribute to $M_D^a$. We can now imagine keeping $D$ fixed and making $a$ smaller; we will then have more vertices inside $D$ and more terms in the sum. In order to take the scaling limit, we need to send the mesh size to zero, but it is not difficult to see that as $a$ tends to zero, the sum defining $M_D^a$ diverges. The scaling limit of the magnetization $M_D^a$ is therefore meaningless.

To understand how to overcome this problem, let's first consider the infinite temperature Ising model. In this case, according to (1), all the spin configurations have the same probability. In other words, the spins behave completely independently of each other (we say that their marginal distributions are independent) and each spin $S_x$ has the same probability $(1/2)$ of being $+1$ or $-1$, independently of all other spins. This means that the magnetization $M_D^a$ is the sum of independent random variables (the spin variables $S_x$), each with the same distribution ($S_x=1$ with probability $1/2$ and $S_x=-1$ with probability $1/2$ for every $x$). Since all variables are independent and equally distributed, we can apply the Central Limit Theorem, one of the most important and useful theorems of probability theory. Because the number of vertices in $D$ is of the order of $(1/a)^2$, the Central Limit Theorem implies that the random variable $aM_D^a$ converges to a normal (Gaussian) random variable as $a\to 0$.

Motivated by this result, let's introduce the *renormalized magnetization*

$$m_D^a = a^{\alpha} \sum_{x\in a\mathbb{Z}^2\cap D} S_x,$$

where $\alpha=\alpha(\beta)$ is a parameter that may depend on the temperature. We've already seen that in the infinite-temperature case ($\beta=0$), the 'correct' choice is $\alpha(0)=1$; any value smaller than $1$ would lead to no limit, while any value larger than $1$ would lead, in the limit, to a very uninteresting random variable identically equal to $0$. The fact that we have to choose $\alpha=1$ tells us something about the fluctuations of the spins, which in this case are normal (or Gaussian) in the limit, as one would expect.

When $\beta>0$, the spin random variables $S_x$ are no longer independent, but as long as $\beta$ is sufficiently small, it turns out that $\alpha=1$ is still the correct choice for the renormalized magnetization, and that the scaling limit still gives a normal random variable [1] (see also [8]). This is true for all values of $\beta<\beta_c=\frac{1}{2}\log(1+\sqrt{2})$. All those values correspond to the high-temperature phase depicted on the right in Figure 5. But what happens at the critical value $\beta_c=\frac{1}{2}\log(1+\sqrt{2})$, corresponding to the middle picture in Figure 5?

Looking at the figure suggests that the fluctuations in the system are very different at the critical point than in the high-temperature regime. According to physics wisdom, the behavior of the renormalized magnetization should differ in an interesting way at the critical point with respect to the high-temperature regime. The scaling limit of the magnetization at the critical point is studied in detail in [5] and [3], where it is proved that, in order to obtain a nontrivial limit (i.e., not identically 0), one needs to choose the value $\alpha(\beta_c)=15/8$. With that choice, the renormalized magnetization has a limit as $a\to 0$:

$$m_D^a = a^{15/8} \sum_{x\in a\mathbb{Z}^2\cap D} S_x \xrightarrow{a\to 0} m_D.$$

The distribution of the limiting random variable $m_D$ is still an open question, but it is known [4] that there is a positive constant $c<\infty$ such that

$$P(m_D>x)\sim e^{-cx^{16}} \quad \text{as } x\to\infty.$$

In particular, this shows that $m_D$ does not have a normal distribution.

It is also shown in [3] that in the scaling limit one can define a sort of magnetization density $\Phi$ such that $m_D=\int_D \Phi(z)\,dz$. The reader should be warned that in writing $\Phi(z)$ I'm committing an abuse of notation since $\Phi$ is not a function and cannot be evaluated pointwise; only its integral over bounded domains or against sufficiently smooth functions of bounded support is well defined. Such objects are called generalized functions or Schwartz distributions (not to be confused with probability distributions).

The magnetization density $\Phi$ has the following property [3]:

$$\int_{[0,sL]^2} \Phi(z)\,dz \overset{\text{dist.}}{=} s^{15/8} \int_{[0,L]^2} \Phi(z)\,dz \qquad (2)$$

where the equality holds in distribution since both sides of the equation are random variables. If $\Phi$ were not present, the integrals would compute the areas of the squares $[0,sL]^2$ and $[0,L]^2$, and the relation would obviously be

$$\int_{[0,sL]^2} dz = s^2 \int_{[0,L]^2} dz,$$

where the exponent $2$ in the $s^2$ factor reflects the two-dimensional nature of the area. Based on this simple observation, equation (2) reveals that the total magnetization behaves like a fractal object with dimension $15/8=2-1/8$. The field $\Phi$, which represents a magnetization density and should therefore have dimension equal to *magnetization/area*, scales with exponent $15/8-2=-1/8$.

A transformation $x\mapsto sx$ like the one considered above is called a *scale transformation*. It belongs to the family of *conformal transformations*, which includes all transformations that preserve angles, at least in a local, 'infinitesimal' sense. Scale transformations, translations and rotations are all transformations that preserve angles globally. Other conformal transformations can be thought of as suitable combinations of those three basic transformations. In the 1970s–80s, some physicists made the very interesting and somewhat mysterious prediction that, after taking a scaling limit, objects like the Ising magnetization at criticality should scale like fractal objects not only under scale transformations, but under all conformal transformations [2,9]. Such objects were named *conformal fields*. The prediction of the appearance of conformal fields in the scaling limit was based on heuristic considerations and could not be proved rigorously. Despite the lack of mathematically rigorous examples, the theoretical study of conformal fields, named Conformal Field Theory (CFT), expanded very rapidly and proved extremely interesting and useful, becoming a central topic in theoretical physics, and producing important applications to various other fields, from statistical mechanics to string theory.

Until recently, however, a rigorous mathematical connection between statistical mechanics and CFT was largely lacking, due to difficulties in the mathematical analysis of scaling limits. The situation started changing at the end of the 1990s, and evolved very rapidly in the first decade of this century. As already mentioned, the scaling limit of the critical Ising magnetization and of the magnetization density $\Phi$ were first shown to exist in [3]. The proof that the Ising magnetization behaves like a fractal object with dimension $15/8$ under all conformal transformations was also proved in [3]. This provides a rigorous example of a conformal field obtained from the critical scaling limit of a model of statistical mechanics. It is also particularly pleasing that the model under consideration is perhaps the most studied model of statistical mechanics, and certainly one that played a crucial role in the development of the theory.

Because of the way they transform under conformal transformations, the critical Ising magnetization and the magnetization density $\Phi$ are called *conformally covariant*. They are also called *Euclidean* fields because they are invariant under translations and rotations, transformations which leave invariant the Euclidean distance, as opposed to *relativistic* fields which are invariant under the *Lorentz transformations* of relativity theory. Their transformation laws are examples of the power laws discussed at the beginning of this paper. The exponents $15/8$ and $1/8$ that appear in those power laws are examples of *critical exponents*.

We started with a general discussion about random fractals and ended with two specific random fractal objects, the renormalized Ising magnetization and its density, whose existence and properties were proved rigorously almost one hundred years after the Ising model was introduced by Lenz. Those years have witnessed the birth and development of important fields of mathematics deeply connected to the Ising model, such as rigorous statistical mechanics, Euclidean field theory and, more recently, *conformal probability*, the study of conformally-invariant and conformally-covariant random objects.

**Acknowledgments**
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**Notes**

1 The discussion above is not restricted to objects that 'live' in a one-dimensional space. One can think, for instance, of covering the unit square with smaller squares of side length $\ell_k = 1/3^k$, or the unit cube with smaller cubes with the same side length. The factor $1/3$ is chosen here for convenience, since the discussion focuses on the triadic Cantor set.

2 The 2016 Nobel Prize in Physics was awarded for the study of certain phase transitions in two dimensions [6].

3 The real picture is more complex, but the microscopic details are not important in this discussion, due to the universality of critical phenomena mentioned earlier.

4 This is somehow related to the fact that the boundary of a segment has the same size (two points) regardless of the size of the segment. In higher dimensions instead, the boundary of square or a (hyper-) cube grows with the size of the square or (hyper-)cube.

5 Note that the symmetry is broken at the level of configurations but not at the distributional level since the system chooses each color with equal probability.

6 The 2008 Nobel Prize in Physics was awarded to Yoichiro Nambu "for the discovery of the mechanism of spontaneous broken symmetry in subatomic physics" and to Makoto Kobayashi and Toshihide Maskawa "for the discovery of the origin of the broken symmetry which predicts the existence of at least three families of quarks in nature".

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