Problemen

| Problem Section

This Problem Section is open to everyone; everybody is encouraged to send in solutions and propose problems. Group contributions are welcome.
For each problem, the most elegant correct solution will be rewarded with a book token worth € 20. At times there will be a Star Problem, to which the proposer does not know any solution. For the first correct solution sent in within one year there is a prize of € 100.
When proposing a problem, please either include a complete solution or indicate that it is intended as a Star Problem.
Please send your submission by e-mail (LaTeX is preferred), including your name and address to problems@nieuwarchief.nl.
The deadline for solutions to the problems in this edition is 1 September 2016.

Problem A (folklore)

Denote for all positive rational numbers x by $f(x)$ the minimum number of 1's needed in a formula for x involving only ones, addition, subtraction, multiplication, division and parentheses. For example, $f(1)=1$, and $f(\frac{1}{3})=4$, as $\frac{1}{3}=\frac{1}{1+1+1}$ and as no such formula exists with at most three 1's. Note that $f(11)\neq 2$ (concatenation of ones is not allowed).
Moreover, denote for all positive rational numbers x by $h_2(x)$ the number $\log_2(p)+\log_2(q)$, where $\log_2$ denotes the base-2 logarithm, and where p, q are positive integers such that $x=\frac{p}{q}$ and $\gcd(p,q)=1$.
Show that for all x, we have

$$f(x) > \tfrac{1}{2}h_2(x).$$

Problem B (folklore)

Suppose that there are $N\geq 2$ players, labeled $1,2,\ldots,N$, and that each of them holds precisely $m\geq 1$ coins of value 1, m coins of (integer) value $n\geq 2$, m coins of value n^2, et cetera. A *transaction* from player i to player j consists of player i giving a finite number of his coins to player j. We say that an N-tuple $(a_1,a_2,\ldots,a_N)$ of integers is (m,n)-*payable* if $\sum_{i=1}^{N} a_i = 0$ and after a finite number of transactions, the i-th player has received (in value) a_i more than he has given away.
Show that for every N-tuple $(a_1,a_2,\ldots,a_N)$ with $\sum_{i=1}^{N} a_i = 0$ to be (m,n)-payable, it is necessary and sufficient that $m > n-\frac{n}{N}-1$.

Problem C (proposed by Wouter Zomervrucht)

For each integer $n\geq 1$ let c_n be the largest real number such that for any finite set of vectors $X\subset\mathbb{R}^n$ with $\sum_{v\in X}|v|\geq 1$ there exists a subset $Y\subseteq X$ with $|\sum_{v\in Y} v|\geq c_n$. Prove the recurrence relation

$$c_1=\tfrac{1}{2}, \qquad c_{n+1}=\tfrac{1}{2\pi n c_n}.$$

Edition 2015-4 We received solutions from Johan Commelin and Raymond van Bommel, Alex Heinis, Pieter de Groen, Alex Heinis, Thijmen Krebs, Hendrik Reuvers and Martijn Weterings.

Problem 2015-4/A (folklore)

Let n be a positive integer. Given a $1\times n$-chessboard made out of paper, one is allowed to fold it along grid lines, and in such a way that the end result is a flat rectangle, say $1\times m$. For example, the following figure shows side views of valid ways of folding a 1×7-chessboard (gray lines depict white squares).

Let a_i for $i = 1, 2, \ldots, m$ be the number of black squares under the i-th square of the resulting rectangle, and consider the tuple $(a_1, a_2, \ldots, a_m)$. So in our examples, the respective corresponding tuples are $(1, 1, 1)$ and $(2, 1, 1)$.
Show that for any positive integer m the m-tuple $(a_1, a_2, \ldots, a_m)$ of non-negative integers can be obtained via the above process if and only if for all $i, j \in \{1, 2, \ldots, m\}$ such that $i + j$ is odd, we have $(a_i, a_j) \neq (0, 0)$.

Solution We received solutions from Pieter de Groen, Thijmen Krebs, Hendrik Reuvers and Martijn Weterings. The book token goes to Martijn Weterings, whose solution the following is based on.
We first show that any tuple $(a_1, a_2, \ldots, a_m)$ obtained by the following process must satisfy $(a_i, a_j) \neq (0, 0)$ for all $i, j \in \{1, 2, \ldots, m\}$ such that $i + j$ is odd.
Draw an arrow facing right on the bottom edge of each square of the $1 \times n$-chessboard. For all $s \in \{1, 2, \ldots, n\}$, define

$$\begin{aligned} b_s &= \begin{cases} 1 & \text{if the } s\text{-th square is white} \\ -1 & \text{if the } s\text{-th square is black} \end{cases} \\ c_s &= \begin{cases} 1 & \text{if the } s\text{-th square is on an even square of the resulting } 1{\times}m\text{–rectangle} \\ -1 & \text{if the } s\text{-th square is on an odd square of the resulting } 1{\times}m\text{–rectangle} \end{cases} \\ d_s &= \begin{cases} 1 & \text{if the arrow on the } s\text{-th square points to the right after folding} \\ -1 & \text{if the arrow on the } s\text{-th square points to the left after folding} \end{cases} \end{aligned}$$

Note that $b_s c_s d_s$ is independent of s, since $b_s b_{s+1}$ is always -1, and exactly one of $c_s c_{s+1}$ and $d_s d_{s+1}$ is -1, depending on whether there is a fold between the s-th square and the $(s+1)$-th square or not.
As there is a connected strip of squares connecting the left edge and the right edge of the resulting $1 \times m$-rectangle, it follows that there is a direction such that above each square of the $1 \times m$-rectangle there is an arrow pointing in that direction. Now suppose for a contradiction that there exist $i, j \in \{1, 2, \ldots, m\}$ such that $i + j$ is odd and $(a_i, a_j) = (0, 0)$. Then there are two white squares s, t above i, j, respectively, that have arrows pointing in the same direction. Hence $b_s = b_t = 1$ and $d_s = d_t$. Moreover, we have $c_s = -c_t$ as $i + j$ is odd. But this contradicts $b_s c_s d_s = b_t c_t d_t$. Hence for all $i, j \in \{1, 2, \ldots, m\}$ such that $i + j$ is odd, we have $(a_i, a_j) \neq (0, 0)$.
Now it remains to show that if $(a_1, a_2, \ldots, a_m)$ is such that for all $i, j \in \{1, 2, \ldots, m\}$ such that $i + j$ is odd, it holds that $(a_i, a_j) \neq (0, 0)$, then it can be obtained via the process described in the problem. We only treat the case that all i with $a_i = 0$ are *even* and that the chessboard starts with a *black* square; the other three cases are similar.
In this case, we are done by the following greedy algorithm.

- Take $n = 2(\sum_{i=1}^{m} a_i) - 1$, and as before, draw an arrow pointing to the right on the bottom edge of each square.
- Place the first (black) square over the first square of the $1 \times m$-rectangle with an arrow pointing to the right.
- Repeatedly fold until there are a_1 black squares lying over the first square, and there is a white square above the second square of the $1 \times m$-rectangle. Note that this is possible as by assumption we have $a_1 \neq 0$. The arrow on the white square is pointing to the right.
- Repeatedly fold until there are a_2 black squares above the second square of the $1 \times m$-rectangle, and there is a black square above the third square of the $1 \times m$-rectangle. The arrow on this square is pointing to the right.
- Repeat the previous two steps alternatingly for the remainder of the squares.

Problem 2015-4/B (proposed by Jinbi Jin)

Let A be a commutative ring with unit, and let I be an ideal of A with $I \neq 0$ and $I^2 = 0$. Let B be the ring of which the elements are triples (a_1, a_2, a_3) where $a_1, a_2, a_3 \in A$ are such that $a_1 + I = a_2 + I = a_3 + I$, with coordinate-wise addition and multiplication. Show that there exist at least four distinct ring homomorphisms $B \to A$.

Solution We received solutions from Johan Commelin and Raymond van Bommel, Alex Heinis and Thijmen Krebs. The book token goes to Johan Commelin and Raymond van Bommel. All received solutions are similar, and the following is based on those.
First note that for $i = 1, 2, 3$, we have the following ring homomorphism.

$$f_i : B \to A,\ (a_1, a_2, a_3) \mapsto a_i$$

Moreover, define

$$g : B \to A,\ (a_1, a_2, a_3) \mapsto a_1 - a_2 + a_3.$$

Note that g is a homomorphism as it is additive, $g(1,1,1)=1$, and for all $(a_1, a_2, a_3), (b_1, b_2, b_3) \in B$ we have that, using that $I^2 = 0$,

$$\begin{aligned} g(a_1,a_2,a_3)\,g(b_1,b_2,b_3) &= (a_1 - a_2 + a_3)(b_1 - b_2 + b_3) \\ &= a_1b_1 - a_2b_2 + a_3b_3 - a_1b_2 - a_2b_1 + a_1b_3 + 2a_2b_2 + a_3b_1 - a_2b_3 - a_3b_2 \\ &= a_1b_1 - a_2b_2 + a_3b_3 + (a_2 - a_1)(b_2 - b_3) + (a_2 - a_3)(b_2 - b_1) \\ &= a_1b_1 - a_2b_2 + a_3b_3 \\ &= g(a_1b_1, a_2b_2, a_3b_3). \end{aligned}$$

Finally, note that these homomorphisms are all distinct, by considering the images of $(i,0,0)$, $(0,i,0)$, $(0,0,i)$ for any non-zero $i \in I$.

Problem 2015-4/C (proposed by Hendrik Lenstra)

Does there exist a non-trivial abelian group A that is isomorphic to its automorphism group?

Solution We received a solution from Alex Heinis. The book token goes to Alex Heinis, whose solution the following is based on.
Let $\mathbb{Z}_3$ denote the ring of 3-adic integers. We show that $A = (\mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}_3$ is isomorphic to its automorphism group.
We will use the following well-known fact about $\mathbb{Z}_3$: the map $\exp : 3\mathbb{Z}_3 \to 1 + 3\mathbb{Z}_3$ defined by $x \mapsto \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ is a group isomorphism (note that the target group is a subgroup of $\mathbb{Z}_3^*$).
We first show that $\mathrm{Aut}(A)$ and $\mathrm{Aut}(\mathbb{Z}_3)$ are isomorphic. Suppose that $\sigma \in \mathrm{Aut}(A)$. As $\mathbb{Z}_3$ has trivial torsion, it follows that $(1,0)$ is the only element in A of order 2. Therefore $\sigma(1,0) = (1,0)$. Moreover, note that 2 is invertible in $\mathbb{Z}_3$, so for all $x \in \mathbb{Z}_3$ we have $\sigma(0,x) = \sigma(0, 2 \cdot \frac{1}{2}x) = 2\sigma(0, \frac{1}{2}x)$, which is an element of $\{0\} \oplus \mathbb{Z}_3$. So any automorphism of A sends $\{0\} \oplus \mathbb{Z}_3$ to itself; this defines a homomorphism $\mathrm{Aut}(A) \to \mathrm{Aut}(\mathbb{Z}_3)$.
This map has an inverse which sends a $\sigma \in \mathrm{Aut}(\mathbb{Z}_3)$ to the automorphism of A given by $(s,x) \mapsto (s, \sigma(x))$. It follows that $\mathrm{Aut}(A)$ is isomorphic to $\mathrm{Aut}(\mathbb{Z}_3)$.
Next, we show that $\mathrm{Aut}(\mathbb{Z}_3)$ and $\mathbb{Z}_3^*$ are isomorphic. Let $\sigma \in \mathrm{Aut}(\mathbb{Z}_3)$. Then first note that for all $x \in \mathbb{Z}$, we have $\sigma(x) = x\sigma(1)$. We claim that $\sigma(x) = x\sigma(1)$ holds for any $x \in \mathbb{Z}_3$.
Note that for any $x \in \mathbb{Z}_3$ and any $e \in \mathbb{Z}_{\geq 0}$, we have $\sigma(3^e x) = 3^e \sigma(x)$ and $\sigma^{-1}(3^e x) = 3^e \sigma^{-1} x$.
It follows that σ preserves the number of factors 3 that occur in elements of $\mathbb{Z}_3$, and therefore in particular that $\sigma(1)$ is invertible in $\mathbb{Z}_3$. So now suppose that $x \in \mathbb{Z}_3$, and write $x = x_0 + x_1 \cdot 3 + x_2 \cdot 3^2 + \cdots$. Then for all $k \in \mathbb{Z}_{>0}$ we have

$$\begin{aligned} \sigma(x) &= \sigma\left(\sum_{i=0}^{k-1} x_i 3^i\right) + 3^k \sigma\left(\sum_{i=k+1}^{\infty} x_i 3^{i-k}\right) \\ &= \left(\sum_{i=0}^{k-1} x_i 3^i\right)\sigma(1) + 3^k \sigma\left(\sum_{i=k+1}^{\infty} x_i 3^{i-k}\right). \end{aligned}$$

It follows that $\sigma(x) = x\sigma(1)$ for all $x \in \mathbb{Z}_3$.
Hence we obtain a homomorphism $\mathrm{Aut}(A) \to \mathbb{Z}_3^*$ sending σ to $\sigma(1)$; this map is an isomorphism with inverse sending $a \in \mathbb{Z}_3^*$ to the automorphism $x \mapsto ax$. Therefore $\mathrm{Aut}(A)$ is isomorphic to $\mathbb{Z}_3^*$.
Now note that $\mathbb{Z}_3^*$ has subgroups $\{\pm 1\}$ and $1 + 3\mathbb{Z}_3$ such that every element $x \in \mathbb{Z}_3^*$ can be written uniquely as sy with $s = \pm 1$ and $y \in 1 + 3\mathbb{Z}_3$. Hence $\mathbb{Z}_3^*$ is isomorphic to $\{\pm 1\} \oplus (1 + 3\mathbb{Z}_3)$; the latter factor is isomorphic to $\mathbb{Z}_3$ via the map $\mathbb{Z}_3 \to 1 + 3\mathbb{Z}_3$, $x \mapsto \exp(3x)$, as desired.