

Problemen

| Problem Section

This Problem Section is open to everyone; everybody is encouraged to send in solutions and propose problems. Group contributions are welcome.

For each problem, the most elegant correct solution will be rewarded with a book token worth €20. At times there will be a Star Problem, to which the proposer does not know any solution. For the first correct solution sent in within one year there is a prize of €100. When proposing a problem, please either include a complete solution or indicate that it is intended as a Star Problem.

Please send your submission by e-mail (LaTeX is preferred), including your name and address to problems@nieuwarchief.nl.

The deadline for solutions to the problems in this edition is 1 March 2016.

Problem A (folklore)

Let $a, b > 1$ be integers such that $2a \leq b$. Does there exist a map

$$f : \{a, a+1, \dots, b\} \rightarrow \{a, a+1, \dots, b\}$$

without fixed points, such that for all $n \in \{a, a+1, \dots, b\}$ we have $f^{f(n)}(n) = n$? Here, for a positive integer k , f^k denotes the k -fold composition

$$\underbrace{f \circ f \circ \dots \circ f}_{k \text{ times}}$$

of f .

Problem B (folklore)

Let $n \geq 3$ be an integer. Two players play the following game. Starting with a sheet of paper with the numbers 1 and 2 on them, the players take turns writing down a new number from 1 to n that is the sum of two numbers already on the sheet. The player who writes down the number n wins.

For which n does the first player have a winning strategy?

Problem C (proposed by Hendrik Lenstra)

Determine all two-sided infinite sequences of positive integers in which each number is the Euler-phi of the next.

Edition 2015-3 We received solutions from Raymond van Bommel, Josephine Buskes, Alex Heinis, Thijmen Krebs, Hendrik Reuvers, Guido Senden, Djurre Tijsma, Steven Troch, Traian Viteam and Robert van der Waall.

Problem 2015-3/A (folklore)

Determine

$$\sum_p \frac{1}{p} \prod_{q < p} \left(1 - \frac{1}{q}\right),$$

where p ranges over all prime numbers, and q ranges over all prime numbers less than p .

Solution We received solutions from Raymond van Bommel, Josephine Buskes, Alex Heinis, Thijmen Krebs, Hendrik Reuvers, Djurre Tijsma, Steven Troch, Traian Viteam and Robert van der Waall. The book token goes to Josephine Buskes, whose solution the following is based on.

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In the following, sums and products always range over prime numbers. We rewrite the given series as a telescoping series:

$$\begin{aligned} \frac{1}{p} \prod_{q < p} \left(1 - \frac{1}{q}\right) &= \left(1 - \left(1 - \frac{1}{p}\right)\right) \prod_{q < p} \left(1 - \frac{1}{q}\right) \\ &= \prod_{q < p} \left(1 - \frac{1}{q}\right) - \prod_{q \leq p} \left(1 - \frac{1}{q}\right). \end{aligned}$$

As $\prod_p \left(1 - \frac{1}{p}\right) = 0$, it then follows that

$$\sum_p \frac{1}{p} \prod_{q < p} \left(1 - \frac{1}{q}\right) = 1.$$

Problem 2015-3/B (proposed by Wouter Zomervrucht)

Let $\mathbb{N} = \{0, 1, \dots\}$ denote the set of natural numbers, and let \mathbb{N}^{2015} denote the set of 2015-tuples $(a(1), a(2), \dots, a(2015))$ of natural numbers. We equip \mathbb{N}^{2015} with the partial order \leq for which $a \leq b$ if and only if $a(k) \leq b(k)$ for all $k \in \{1, 2, \dots, 2015\}$. We say that a sequence a_1, a_2, \dots in \mathbb{N}^{2015} is good if for all $i < j$ we have $a_i \not\leq a_j$.

– Show that all good sequences are finite.

We say that a sequence a_1, a_2, \dots in \mathbb{N}^{2015} is perfect if it is good and for all i and for all $k \in \{1, 2, \dots, 2015\}$ we have $\{a_i(k) \leq 2015i$.

– Does there exist a positive integer N such that all perfect sequences have length at most N ?

Solution We received solutions from Raymond van Bommel, Alex Heinis and Guido Senden, and a solution of the first part from Djurre Tijsma. The book token goes to Alex Heinis, whose solution the following is based on.

For the first part, first note that every infinite sequence x_1, x_2, \dots of natural numbers has a non-decreasing subsequence; indeed, we can define it recursively by setting k_1 to be the minimal index for which x_{k_1} is minimal, and by setting k_{i+1} to be the minimal index larger than k_i for which $x_{k_{i+1}} = \min_j \{x_j : x_j \geq x_{k_i}\}$ for all i .

So suppose for a contradiction that there exists an infinite good sequence a_1, a_2, \dots in \mathbb{N}^{2015} . By passing to a subsequence at most 2015 times, we obtain an infinite good sequence a'_1, a'_2, \dots in \mathbb{N}^{2015} such that $a'_i(k) \leq a'_j(k)$ for all i, j, k , i.e. $a'_i \leq a'_j$ for all i, j ; a contradiction. Therefore no infinite good sequence can exist.

For the second part, suppose for a contradiction that for every positive integer N there exists a perfect sequence of length at least N . Since every good sequence is finite by the first part, this implies that there are infinitely many perfect sequences. Note that in any perfect sequence a_1, a_2, \dots , the i -th term is an element of the finite set $\{1, 2, \dots, 2015i\}^{2015}$.

By the pigeonhole principle, we can now recursively define a_i to be an element of $\{1, 2, \dots, 2015i\}^{2015}$ such that there are infinitely many perfect sequences that start with a_1, a_2, \dots, a_i . This defines an infinite sequence a_1, a_2, \dots . This sequence is good; for all $i < j$ we note that a_1, a_2, \dots, a_j occurs as the first j terms of a perfect sequence, so $a_i \not\leq a_j$. But this contradicts the non-existence of infinite good sequences obtained in the first part.

Problem 2015-3/C (proposed by Marcel Roggeband)

The (first) Bernoulli numbers B_n for integers $n \geq 0$ are defined by the following recursive formula.

$$\begin{aligned} B_0 &= 1, \\ B_n &= -\sum_{i=0}^{n-1} \binom{n}{i} \frac{B_i}{n-i+1} \quad \text{for } n > 0. \end{aligned}$$

Show that the Bernoulli numbers satisfy the following identity for all $n > 1$

$$B_n = n! \sum_{i=1}^n \sum_{\sigma} \frac{(-1)^{i-1}}{\sigma_1! \dots \sigma_i!} \left(\frac{1}{2} - \frac{1}{\sigma_i + 1}\right)$$

In this sum, σ runs through all ordered i -tuples $(\sigma_1, \dots, \sigma_i)$ of integers such that $\sigma_1 + \dots + \sigma_i = n + i - 1$ and $\sigma_j \geq 2$ for all j .

Solution We received solutions from Raymond van Bommel, Alex Heinis, Thijmen Krebs and Hendrik Reuvers. The book token goes to Hendrik Reuvers, whose solution the following is based on.

Write $C_n = \frac{B_n}{n!}$, so that the C_k satisfy $C_0 = 1$ and

$$C_n = - \sum_{i=0}^{n-1} \frac{C_i}{(n-i+1)!}.$$

Moreover, write

$$A_n = \sum_{i=1}^n \sum_{\sigma} \frac{(-1)^{i-1}}{\sigma_1! \dots \sigma_i!} \left(\frac{1}{2} - \frac{1}{\sigma_i + 1} \right)$$

where in the sum, σ runs through all ordered i -tuples $(\sigma_1, \dots, \sigma_i)$ of integers such that $\sigma_1 + \dots + \sigma_i = n + i - 1$ and $\sigma_j \geq 2$ for all j .

We are done once we show that $A_n = C_n$ for all $n \geq 2$. We do so by induction on n . For $n = 2$, we have $A_n = \frac{1}{12} = C_n$. So suppose that for some $m \geq 2$ we have $A_k = C_k$ for all k such that $2 \leq k \leq m$. We then show that $A_{m+1} = C_{m+1}$.

$$\begin{aligned} C_{m+1} &= - \sum_{k=0}^m \frac{C_k}{(m-k+2)!} = \frac{-1}{(m+2)!} + \frac{\frac{1}{2}}{(m+1)!} + \sum_{k=2}^m \frac{-A_k}{(m-k+2)!} \\ &= \frac{1}{(m+1)!} \left(\frac{1}{2} - \frac{1}{m+2} \right) + \sum_{k=2}^m \sum_{i=1}^k \sum_{\sigma} \frac{(-1)^i}{\sigma_1! \dots \sigma_i!} \frac{1}{(m-k+2)!} \left(\frac{1}{2} - \frac{1}{\sigma_i + 1} \right), \end{aligned}$$

where in the sum, σ runs through all ordered i -tuples $(\sigma_1, \dots, \sigma_i)$ of integers such that $\sigma_1 + \dots + \sigma_i = k + i - 1$ and $\sigma_j \geq 2$ for all j .

Note that the first term in the expression obtained for C_{m+1} equals the first term of A_{m+1} . Moreover, for $i \geq 1$, the (k, i, σ) -term in the expression obtained for C_{m+1} equals the $(i+1, \sigma')$ -term of A_{m+1} , where $\sigma' = (m-k+2, \sigma_1, \dots, \sigma_i)$; this tuple is admissible since $(m-k+2) + (k+i-1) = (m+1) + (i+1) - 1$. The corresponding map $(k, i, \sigma) \mapsto (i+1, \sigma')$ on index sets is a bijection (with inverse given by $(j, \sigma') \mapsto (m+2-\sigma'_1, j-1, \sigma)$ where $\sigma = (\sigma'_2, \dots, \sigma'_j)$), as desired.

Rectification

In the list of solvers of 2015-2/A, Hendrik Reuvers was missing, but should have been there. We apologise for this omission.

