François Genoud
Delft Institute of Applied Mathematics
Delft University of Technology
s.f.genoud@tudelft.nl

Column Tenure-tracker

Bifurcations in nonlinear PDEs

In this column holders of a tenure track position introduce themselves. The tenure track positions in mathematics became available in 2013. Excellent researchers could apply in several expertise areas of mathematics. François Genoud has a tenure track position at Delft University of Technology.

I am a Swiss mathematician from Lausanne, where I did my undergraduate studies in physics and my PhD in mathematics with Charles A. Stuart. Since January 2015 I am an Assistant Professor Tenure Track in the Analysis Group of Delft University of Technology. I arrived here after a six-year postdoctoral journey through Oxford, Edinburgh and Vienna, which allowed me to diversify my research interests in important areas of mathematical physics.

My research lies in the rigorous mathematical analysis of differential equations. An important part of my work revolves around bifurcation theory, which is a powerful tool to understand qualitative properties of nonlinear partial differential equations. Partial differential equations (PDEs) are a natural language to describe many physical phenomena. The solutions of the equations represent physical quantities characterising the state of a given system. Bifurcation theory explains how the possible states of the system change when some physical parameters are varied.

The core of my work is in the analysis of nonlinear PDEs. I develop and apply abstract functional analytic methods (e.g. topological degree theory, min-max methods from the calculus of variations, implicit function theorems) to study PDEs in a rigorous mathematical framework. Thanks to my early education in physics, I am always keen on understanding the underlying physical models as well. Many important phenomena in nature involve some sort of oscillatory motion, modelled by 'wave equations' that are typically nonlinear. Even though it is in general not possible to solve the equations explicitly, the mathematical analyst can prove theorems about existence and properties (regularity, stability, blow-up, et cetera) of the solutions. This is essential for a deep understanding of the physical theories formulated through the equations.

I have applied nonlinear analysis to various important PDEs coming from mathematical physics, for instance in models of large-scale oceanic waves based on the Euler equation [7], or the study of phase transitions in nematic liquid crystals [1]. A large part of my research has been concerned with nonlinear Schrödinger (NLS) equations, which arise in the modelling of a variety of wave motions, including the propagation of light in optical fibres, Langmuir waves in plasma, Bose–Einstein condensates, or water waves on the sea. For NLS equations, I have proved the existence of stable nonlinear waves, known as 'solitons'. These

are idealisation of waves encountered in real-world systems, characterised by strong localisation in space (and/or time) and strong stability properties. Such waves can for instance represent narrow laser/light beams in nonlinear optical media, solitary waves on a water surface, rogue waves, et cetera.

Nonlinear wave guides

Bifurcation theory has proved especially useful to study equations having a nontrivial spatial dependence, sometimes referred to as *inhomogeneous NLS*. In the context of a planar nonlinear waveguide, they take the general form

$$\begin{aligned} &i\partial_z\psi + \partial^2_{xx}\psi + f(x,|\psi|^2)\psi = 0, \\ &\psi = \psi(x,z) : \mathbb{R}\times\mathbb{R} \to \mathbb{C}, \end{aligned} \tag{1}$$

where z is the direction of propagation of the wave and $\partial^2_{xx}\psi$ is the Laplacian of the (complex envelope of the) electric field ψ in the transverse direction x. In this model, the nonlinear response $f(x,|\psi|^2)$ represents the electric permittivity of the material. In *self-focusing* media, this is a positive function, increasing in the field intensity $|\psi|^2$. A laser beam travelling in the material locally modifies its permittivity, thereby focusing itself along the propagation axis $x = 0$. The dependence on x accounts for inhomogeneities in the medium. The most commonly used materials are the *Kerr media*, for which $f(x,|\psi|^2) = V(x)|\psi|^2$, for some $V : \mathbb{R} \to \mathbb{R}_+$.

We call *soliton* a standing wave solution of the form $\psi(x,z) = u(x)\mathrm{e}^{ikz}$, where $k \in \mathbb{R}$ and $u : \mathbb{R} \to \mathbb{R}$ is localized — typically $u \in H^1(\mathbb{R})$ and $u(x) \to 0$ exponentially as $|x| \to \infty$. Such a solution of (1) exists if and only if u satisfies the nonlinear ordinary differential equation

$$u''(x) + f(x,u^2(x))u(x) = ku(x), \quad u \in H^1(\mathbb{R}). \tag{2}$$

Soliton curves $k \mapsto \psi_k(x,z) = u_k(x)\mathrm{e}^{\mathrm{i}kz}$ can be obtained by bifurcation techniques applied to (2). Heuristically, the existence of solitons is allowed by a balance in (1) between the diffraction modelled by the Laplacian and the self-focusing effects due to the nonlinear term $f(x,|\psi|^2)\psi$. Their stability then depends on the monotonicity of the function $k \mapsto \|u_k\|_{L^2}$ and on the spectral properties of linearised operators associated with (1)–(2).

The combination of space-dependent coefficients and nonlinearities more general than the pure-power law $f(x,|\psi|^2) = |\psi|^{p-1}$ $(p > 1)$ is of major interest for applications, but has only been scarcely investigated in the mathematical literature. I have obtained curves $k \mapsto u_k \in H^1(\mathbb{R})$ of stable solitons for a nonlinear response of the form

$$f(x,|\psi|^2) = V(x)|\psi|^{p-1} \quad \text{or}$$
$$f(x,|\psi|^2) = V(x)\frac{|\psi|^{p-1}}{1+|\psi|^{p-1}} \quad (1 < p < 5)$$

under appropriate regularity and decay assumptions on the coefficient $V : \mathbb{R} \to \mathbb{R}$, see [2–4]. Another model of interest in nonlinear optics is given by

$$f(x,|\psi|^2) = \epsilon\delta(x) + 2|\psi|^2 - |\psi|^4,$$

where $\epsilon > 0$ is a coupling constant and the Dirac mass $\delta(x)$ models a narrow attractive potential centred at $x = 0$. A remarkable feature of this model is that explicit solutions are available, that can be expressed in terms of elementary functions. Their stability can be proved by bifurcation and spectral analysis [5].

Wave collapse

What is meant here by stability is that, given an 'initial condition' at $z = 0$, $\psi(\cdot,0) \in H^1(\mathbb{R})$, close to the initial condition u_k of the standing wave $\psi_k(x,z) = u_k(x)\mathrm{e}^{ikz}$, the corresponding solution $\psi(x,z)$ of (1) remains close (in an appropriate sense) to $\psi_k(x,z)$, for all $z > 0$. In particular $\psi(x,z)$ exists for all $z > 0$. However, in some situations, the focusing effects will beat the diffraction in the dynamics of (1), giving rise to solutions which *blow up* at a finite propagation distance $Z > 0$ in the waveguide, in the sense that

$$\lim_{z\uparrow Z} \|\partial_x\psi(x,z)\|_{L^2} = \infty.$$

This phenomenon of 'wave collapse' — where, typically, all the beam power concentrates on the axis of propagation at the blow-up point — has been well-known since the early days of nonlinear optics. Of course, the collapse indicates that the physical relevance of the model breaks down at the blow-up point. However, the dynamics leading to the blow-up give valuable information on the behaviour of the beam undergoing intense self-focusing. The formation of singularities in NLS equations has attracted substantial interest in the past twenty years, but mostly in the pure-power case, $f(x,|\psi|^2) = |\psi|^{p-1}$. I have contributed to extend the theory to inhomogeneous NLS equations [6], and further work in this direction is in progress with Elek Csobo, my PhD student.

References

1 S. Bachmann and F. Genoud, Effective theories for liquid crystals and the Maier–Saupe phase transition, under review, preprint arxiv.org/abs/1508.05025.

2 F. Genoud, Existence and orbital stability of standing waves for some nonlinear Schrödinger equations, perturbation of a model case, *J. Differential Equations* 246 (2009), 1921–1943.

3 F. Genoud, Bifurcation and stability of travelling waves in self-focusing planar waveguides, *Adv. Nonlinear Stud.* 10 (2010), 357–400.

4 F. Genoud, Orbitally stable standing waves for the asymptotically linear one-dimensional NLS, *Evolution Equations and Control Theory* 2 (2013), 81–100.

5 F. Genoud, B.A. Malomed and R.M. Weishäupl, Stable NLS solitons in a cubic-quintic medium with a delta-function potential, to appear in *Nonlinear Anal.*, preprint arxiv.org/abs/1409.6511.

6 F. Genoud and V. Combet, Classification of minimal mass blow-up solutions for an L^2 critical inhomogeneous NLS, to appear in *J. Evol. Equ.*, preprint arxiv.org/abs/1503.08915.

7 F. Genoud and D. Henry, Instability of equatorial water waves with an underlying current, *J. Math. Fluid Mech.* 16 (2014), 661–667.