

Problemen

Problem Section

This Problem Section is open to everyone; everybody is encouraged to send in solutions and propose problems. Group contributions are welcome.

For each problem, the most elegant correct solution will be rewarded with a book token worth €20. At times there will be a Star Problem, to which the proposer does not know any solution.

For the first correct solution sent in within one year there is a prize of €100.

When proposing a problem, please either include a complete solution or indicate that it is intended as a Star Problem. Electronic submissions of problems and solutions are preferred (problems@nieuwarchief.nl).

The deadline for solutions to the problems in this edition is 1 March 2015.

Problem A (proposed by Jan Turk)

Let $k > 3$ be an integer. Determine the variance of the greatest common divisor of k positive integers. Here we mean the limit, as $n \rightarrow \infty$, of the variance of the greatest common divisor of k integers in $\{1, 2, \dots, n\}$ with respect to the uniform distribution on $\{1, \dots, n\}^k$.

Problem B (folklore)

The evil Eve has locked Alice and Bob in a room without windows. Outside the room, there is a corridor with 64 doors. Eve puts a key behind one of the doors and a crocodile behind each of the others. Then she hangs up a light bulb above each of the doors, and for each light bulb, switches it on or off. Then Eve brings Alice into the corridor, tells her which door hides the key and tells her to choose one of the light bulbs and change the state of that chosen light bulb. After Alice leaves, Eve brings Bob to the corridor, and tells him to open a door of his own choice. Alice and Bob are allowed to discuss a strategy before Alice is shown where the key is, but not after.

- Give a strategy that guarantees Bob to find the key.
- For which positive integers n does such a strategy exist if there are n doors?

Problem C (folklore)

Let X_3 be the collection of three-element subsets of $\{1, 2, \dots, 8\}$, and let X_4 be the collection of four-element subsets of $\{1, 2, \dots, 11\}$. Does there exist an injective map $\phi: X_3 \rightarrow X_4$ with the following properties?

- For all subsets $V \subseteq X_3$, we have $\#(\bigcap_{v \in V} \phi(v)) \geq \#(\bigcap_{v \in V} v)$.
- For all $v, v' \in X_3$, if $v \cap v' = \emptyset$, then $\phi(v) \cap \phi(v') = \emptyset$.

Edition 2014-2 We received solutions from Rik Bos (Bunschoten), Pieter de Groen (Brussels), Alex Heinis (Amsterdam), Nicky Hekster (Amstelveen), Carlo Pagano (Leiden) and Rob van der Waall (Huizen).

Problem 2014-2/A (proposed by Wouter Zomervrucht)

Let n be a positive integer. Let M be an $n \times n$ -matrix with entries in $\{1, 2, \dots, n\}$. Let r be the complex eigenvalue with the largest absolute value. Show that $n \leq |r| \leq n^2$.

Solution We received correct solutions from Rik Bos, Pieter de Groen, Alex Heinis, Carlo Pagano and Rob van der Waall. The book token is awarded to Rob van der Waall.

Let $\|\cdot\|$ be the Frobenius matrix norm (i.e. for $M = (m_{ij})_{i,j=1}^n$, we have $\|M\|^2 = \sum_{i=1}^n \sum_{j=1}^n |m_{ij}|^2$). Gelfand's formula states

$$|r| = \lim_{k \rightarrow \infty} \|M^k\|^{1/k}.$$

Let N be another matrix as in the problem, with largest eigenvalue s . Suppose $M \leq N$,

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Oplossingen

Solutions

i.e. inequality on all entries, then also $M^k \leq N^k$, so $\|M^k\| \leq \|N^k\|$, and $r \leq s$. Let M_0 be the $n \times n$ -matrix with all entries equal to 1. It has eigenvalues 0 and n . Note that nM_0 has eigenvalues 0 and n^2 . We have $M_0 \leq M \leq nM_0$, hence $n \leq r \leq n^2$ as desired.

Problem 2014-2/B (proposed by Hans Zwart)

Let X be a unital \mathbb{R} -algebra with multiplicative unit 1, and let $\|\cdot\|$ be a *submultiplicative norm* on X , i.e. a map $\|\cdot\|: X \rightarrow \mathbb{R}$ satisfying the following properties:

- $\|1\| = 1$;
- if $x \in X$ satisfies $\|x\| = 0$, then $x = 0$;
- for all $a \in \mathbb{R}, x \in X$, we have $\|ax\| = |a| \|x\|$;
- for all $x, y \in X$, we have $\|x + y\| \leq \|x\| + \|y\|$ and $\|xy\| \leq \|x\| \|y\|$.

Let $C: \mathbb{R} \rightarrow X$ be a map such that $C(0) = 1$ and such that for all $s, t \in \mathbb{R}$, we have

$$2C(s)C(t) = C(s + t) + C(s - t).$$

Suppose that

$$\sup_{s \in \mathbb{R}} \|C(s) - 1\| < \frac{3}{2}.$$

Show that $C = 1$.

Solution We received no correct solutions. The following solution is based on that of the proposer, Hans Zwart.

Let $t \in \mathbb{R}$. Then we have

$$C(2t) - 1 = 2C(t)^2 - C(0) - 1 = 2C(t)^2 - 2 = 2(C(t) - 1)^2 + 4(C(t) - 1).$$

Hence for all $t \in \mathbb{R}$, we have $4(C(t) - 1) = (C(2t) - 1) - 2(C(t) - 1)^2$, so if $\rho = \sup_{s \in \mathbb{R}} \|C(s) - 1\|$, then for all $t \in \mathbb{R}$, we have

$$\begin{aligned} 4\|C(t) - 1\| &= \left\| (C(2t) - 1) - 2(C(t) - 1)^2 \right\| \\ &\leq \|C(2t) - 1\| + 2\|C(t) - 1\|^2 \\ &\leq \rho + 2\rho^2, \end{aligned}$$

hence $4\rho \leq \rho + 2\rho^2$, or equivalently, $\rho(2\rho - 3) \geq 0$. As we assumed that $\rho < \frac{3}{2}$, it follows that $\rho = 0$, hence $C(t) = 1$ for all $t \in \mathbb{R}$, as desired.

Problem 2014/2-C (proposed by Hendrik Lenstra)

Let G be a finite group. Let n be the number of automorphisms σ of G such that for all $x \in G$, the element $\sigma(x)$ is conjugate to x . Show that every prime divisor of n divides the order of G .

Solution We received solutions from Rik Bos, Alex Heinis, Nicky Hekster, Carlo Pagano and Rob van der Waall. All their solutions were along the same lines, which we reproduce here. The book token goes to Alex Heinis.

Let A be the set of $\sigma \in \text{Aut}(G)$ such that for all $x \in G$, the element $\sigma(x)$ is conjugate to x , so $n = \#A$. Let p be a prime divisor of n . Note that A is a subgroup of $\text{Aut}(G)$, hence by Cauchy's theorem has an element σ of order p . This element σ acts on each conjugacy class of G , and the orbits have length 1 or p .

Let $H = \{x \in G : \sigma(x) = x\}$ be the union of the orbits of length 1, which is a proper subgroup of G .

Lemma. *Let $H \subset G$ be a subgroup with $H \neq G$. Then the union of the conjugates of H is not G .*

Oplossingen

Solutions

Proof. The group H has at most $[G : H]$ conjugates, because gHg^{-1} depends only on $gH \in G/H$. These conjugates all have $\#H$ elements and have the element 1 in common, hence the union of the conjugates has fewer than $[G : H] \cdot \#H = \#G$ elements. \square

By the lemma, we see that there is an element of $x \in G$ that is not conjugate to any element of H . Its conjugacy class C is a union of orbits for $\langle \sigma \rangle$, which all have length p as C is disjoint with H , so $\#C$ is divisible by p .

By the orbit-stabilizer theorem, the order of G is divisible by the length of the conjugation orbit C of x . \square

