Peter J.C. Dickinson
Faculty of Electrical Engineering, Mathematics and Computer Science
University of Twente
p.j.c.dickinson@utwente.nl

Column Tenure-tracker

The copositive cone

In this column holders of a tenure track position introduce themselves. The tenure track positions in mathematics became available in 2013. Excellent researchers could apply in several expertise areas of mathematics. Peter J.C. Dickinson has a tenure track position at the University of Twente.

In April 2014 I started a tenure track position at the University of Twente, partially funded by NWO. Due to this I was invited to write a brief article introducing myself to the readers of *Nieuw Archief voor Wiskunde*.

Starting naturally enough at the beginning: I grew up in a village on the outskirts of London called Croxley Green. I then later went to the University of Cambridge to study mathematical physics. After finishing my masters I decided to take a break and did some backpacking around Australia. Well I say backpacking, but actually I spent most of my time in a single village in Western Australia doing scuba diving.

What brought me back to Europe, and ultimately academia, was a girl. I met my partner in Australia and I decided to move to the Netherlands to be with her. I figured a job in scuba diving would not be quite as exciting in the Netherlands, so instead I returned to academia. I was fortunate enough to be given the opportunity by Prof. Dr. M. Dür to do a PhD with her at the University of Groningen, starting in May 2009. After four years I finished my PhD (with a cum laude distinction) and started a PostDoc at the University of Vienna, with Prof. Dr. I.M. Bomze. This PostDoc was cut short due to the tenure track position that I started at the University of Twente in April 2014.

I think that this is enough personal details to be getting on with. Being a mathematician writing in a mathematical magazine, I think it is time I got on to what we are all actually interested in, my research. The main focus of my research is currently the copositive cone, and in particular its use in connection with optimisation. A symmetric matrix A is defined to be copositive if $\mathbf{x}^T A\mathbf{x} \geq 0$ for all nonnegative vectors $\mathbf{x}$. The set of copositive matrices is then

$$\text{Copositive Cone,} \quad \mathcal{COP}^n := \{A \in \mathcal{S}^n \mid \mathbf{x}^T A\mathbf{x} \geq 0 \text{ for all } \mathbf{x} \in \mathbb{R}^n_+\},$$

where $\mathcal{S}^n$ is the set of symmetric matrices and $\mathbb{R}^n_+$ is the set of nonnegative vectors.

Although the definition of this cone is fairly simple, its properties are far from simple. In fact even checking if a matrix is copositive is an NP-hard problem [7]. A problem being difficult can sometimes scare you away from it, but can also be an incentive to study it further. With copositivity the latter has been true for me, along with many others.

Copositivity was first developed within the field of linear algebra, with the concept being introduced by Prof. Dr. T.S. Motzkin in the 1950s [6]. One reason for studying it is its innate beauty, however as much as we like to discover the beauty in mathematics, society normally asks for some applications. For copositivity, important applications were provided around the turn of the millennium through conic optimisation [1–2]. Conic optimisation is a type of convex optimisation where we minimise or maximise a linear function over the intersection of a

closed convex cone and an affine space. When considering this for the copositive cone we call it copositive optimisation. In other words a copositive minimisation problem is one of the form

$$\begin{aligned} \min \quad & \mathrm{tr}(CX) \\ \text{s.t.} \quad & \mathrm{tr}(A_i X) = b_i \quad \text{for all } i = 1, \dots, m, \\ & X \in \mathcal{COP}^n, \end{aligned}$$

where "tr" denotes the trace, $b_1, \dots, b_m \in \mathbb{R}$ and $C, A_1, \dots, A_m \in \mathcal{S}^n$.

Normally we would expect convex optimisation problems to be somewhat easy (i.e. solvable in polynomial time) [5]. However amazingly, although copositive optimisation is a form of convex optimisation, it can be used to give exact formulations of some NP-hard discrete optimisation problems. An example of such a problem that can be formulated in this way is that of finding the stability number of a graph [8]. These applications put copositive optimisation on the boundary between continuous and discrete optimisation, as well as meaning that copositive optimisation problems cannot in general be solved in polynomial time (unless P=NP).

The advantage of copositive optimisation is that it allows us to apply techniques from convex optimisation to some NP-hard discrete problems. It is hoped that through understanding copositive optimisation we may be able to develop a one size fits all approach to solving these problems. Even if this lofty aim is not achieved, studying copositivity will still provide new insights and methods. We can also expand on results developed in connection to copositivity in order to consider a greater class of problems [3–4, 9].

A form of conic optimisation that is regularly used in the real world (in the background) is linear optimisation. Another form of conic optimisation that is on the brink of breaking out of academia is semidefinite optimisation. In the future I believe that copositive optimisation will also become highly important, making a (co-)positive contribution to our lives.

References

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