Problemen

Problem Section

This Problem Section is open to everyone; everybody is encouraged to send in solutions and propose problems. Group contributions are welcome.
For each problem, the most elegant correct solution will be rewarded with a book token worth €20. At times there will be a Star Problem, to which the proposer does not know any solution. For the first correct solution sent in within one year there is a prize of €100.
When proposing a problem, please either include a complete solution or indicate that it is intended as a Star Problem. Electronic submissions of problems and solutions are preferred (problems@nieuwarchief.nl).
The deadline for solutions to the problems in this edition is 1 June 2014.

Problem A (proposed by Hendrik Lenstra)

Let G be a group, and let $a, b \in G$ be two elements satisfying $\{gag^{-1} : g \in G\} = \{a, b\}$. Prove that for all $c \in G$ one has $abc = cba$.

Problem B (the attribution will appear in the September issue of 2014)

Let K be a field, and consider for all positive integers n the subset S_n of $x \in K^*$ that can be written as the sum of n squares in K. Show that the subgroup of K^* generated by S_n is equal to $S_{t(n)}$. Here, for a positive integer n, we denote by $t(n)$ the smallest power of two that is greater than or equal to n.

Problem C (folklore)

Given five pairwise distinct points A, B, C, D, E in the plane, no three of which are collinear, and given a line l in the plane not passing through any of the five points. Assume that l intersects the conic section c passing through A, B, C, D, E. Construct the intersection points of l and c.
One of the solutions which uses the smallest number of moves will be awarded the book token.
Here, given a collection of points, lines, and circles in the plane, a *move* consists of adding to the collection either a line through two of the points, or a circle centered at one of them and passing through another. At any time one is allowed to freely add any intersection point among the lines and circles, as well as any sufficiently general point, either in the plane, or on any of the lines or circles.
For example, given a line ℓ and a point P on ℓ one can construct a line through P and perpendicular to ℓ in three moves as follows. Choose a point M not on ℓ. For the first move, take the circle C centered at M and going through P. Let Q be the second point of intersection between C and ℓ. For the second move, add the line through Q and M and let R be the second point of intersection between this line and C. Finally, add the line PR, which is perpendicular to ℓ.

Edition 2013-3 We received solutions from Pieter de Groen (Brussels), Alex Heinis (Amsterdam), Tejaswi Navilarekallu and Sander Zwegers (Keulen).

Problem 2013-3/A (proposed by Hendrik Lenstra)

Let $a, b \in \mathbb{C}$. Show that if there exists an irreducible polynomial $f \in \mathbb{Q}[X]$ such that $f(a) = f(a+b) = f(a+2b) = 0$, then $b = 0$.

Solution We received solutions from Alex Heinis, Tejaswi Navilarekallu and Sander Zwegers. The following solution is based on that of Alex Heinis, who will receive the book token.
Let S be the (finite) set of complex zeros of f and let H be the convex hull of S, which is a polygon or a line segment. Let $v \in S$ be a vertex of H.
Since f is irreducible, the Galois group G of its splitting field acts transitively on S, hence there is an element $\sigma \in G$ with $\sigma(a+b) = v$.
Now $\sigma(a)$ and $\sigma(a+2b)$ are also in S, and $v = \sigma(a+b)$ lies on the line segment connecting them. This contradicts the fact that v is a vertex, unless $b = 0$.

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Solutions

Problem 2013-3/B (a result due to B. Konstant and N. Wallach [1])
Let n be a positive integer, and let e_{ij} be an integer for all $1 \leq j \leq i \leq n$. Show that there exists an $n \times n$–matrix with entries in $\mathbb{Z}$ such that the eigenvalues of the top left $i \times i$–minor are $e_{i1}, \ldots, e_{ii}$ (with multiplicity).

Solution We received correct solutions from Pieter de Groen, Alex Heinis and Sander Zwegers. All gave a solution similar to the one we present below. The book token goes to Sander Zwegers. For any square matrix A we denote the top left $i \times i$–minor by $m_i(A)$. We also use the notation $\chi_A(\lambda) := \det(\lambda I - A)$ for the characteristic polynomial of A. Let S_n be the set of all $n \times n$–matrices $A = (a_{ij})$ over the polynomial ring $\mathbb{Z}[\lambda]$ such that $a_{i+1,i} = 1$ for all $1 \leq i < n$ and such that $a_{i,j} = 0$ if $i \geq j+2$, i.e., A has ones on the line just below its main diagonal and has zeroes below this line of ones. (We allow λ in the matrices because we will work with characteristic polynomials.) It suffices to prove the following claim:

Claim. *Let n be a positive integer and let e_{ij} be an integer for all $1 \leq j \leq i \leq n$. Then there exists a matrix $A \in S_n$ with entries in $\mathbb{Z}$ such that $\chi(m_i(A)) = (\lambda - e_{i1}) \cdots (\lambda - e_{ii})$ for all $1 \leq i \leq n$.*

We prove this claim by induction on n. The base case $n = 1$ is trivial: the matrix (e_{11}) will do. Now suppose that the claim is true for some $n = N-1 \geq 1$ and let us prove it for $n = N$. So let e_{ij} be an integer for all $1 \leq j \leq i \leq N$ and let $A \in S_{N-1}$ be the matrix we get from the claim for $n = N-1$ and the integers e_{ij} with $1 \leq j \leq i \leq N-1$. Since $1, \chi(m_1(A)), \ldots, \chi(m_{N-1}(A)), \lambda\chi(m_{N-1}(A))$ are monic polynomials over $\mathbb{Z}$ of degree $0, 1, \ldots, N-1, N$ (respectively), they form a $\mathbb{Z}$–basis for the abelian group of polynomials of degree $\leq N$ with integer coefficients. So there exist integers $x_1, \ldots, x_{N+1}$ such that the degree–N polynomial $(\lambda - e_{N1}) \cdots (\lambda - e_{NN})$ is equal to the linear combination

$$-x_1 + (-1)^{N+1} x_{N+1} \lambda \chi(m_{N-1}(A)) + \sum_{i=2}^{N} (-1)^i x_i \chi(m_{i-1}(A)).$$

Since this polynomial is monic of degree N, we know that $x_{N+1} = (-1)^{N+1}$. We claim that we can take the following matrix to prove the claim for $n = N$:

$$B = \begin{pmatrix} & & & x_1 \\ & A & & x_2 \\ & & & \vdots \\ 0 & \cdots & 1 & x_N \end{pmatrix}.$$

First of all, $A \in S_{N-1}$ implies that $B \in S_N$. Moreover, for $i < N$ we have $\chi(m_i(B)) = \chi(m_i(A)) = (\lambda - e_{i1}) \cdots (\lambda - e_{ii})$ by the construction of B and the choice of A. So we are left to compute $\chi(m_N(B)) = \chi(B)$.

Lemma. *For every $M = (M_{ij}) \in S_n$ ($n \geq 2$) we have*

$$\det(M) = (-1)^n \left(-M_{1,n} + \sum_{i=2}^{n} (-1)^i M_{i,n} \det(m_{i-1}(M)) \right).$$

Proof. Expand the determinant with respect to the last row to find that

$$\det(M) = M_{n,n} \det(m_{n-1}(M)) - \det \begin{pmatrix} & & & M_{1,n} \\ & m_{n-2}(M) & & M_{2,n} \\ & & & \vdots \\ 0 & \cdots & 1 & M_{n-1,n} \end{pmatrix}.$$

Now use induction and the fact that $m_i(m_j(M)) = m_i(M)$ for all $i \leq j \leq n$. □

The lemma enables us to compute the characteristic polynomial of B:

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$$\begin{aligned}\chi(B) &= (-1)^N \det(B-\lambda I) = -x_1 + \sum_{i=2}^{N}(-1)^i(B-\lambda I)_{i,N}\det(m_{i-1}(B-\lambda I))\\ &= -x_1 + x_2\chi(m_1(A)) + \cdots + (-1)^{N-1}x_{N-1}\chi(m_{N-2}(A))\\ &\quad + (-1)^N(x_N-\lambda)\chi(m_{N-1}(A)).\end{aligned}$$

By definition of $x_1, \ldots, x_N$, this polynomial is equal to $(\lambda - e_{N1})\cdots(\lambda - e_{NN})$ and this concludes the proof of the claim for $n = N$.

Problem 2013-3/C (proposed by Bart de Smit and Hendrik Lenstra)
Let A be a finite commutative unital ring. Does there exist a pair (B, f) with B a finite commutative unital ring in which every ideal is principal, and f an injective ring homomorphism $A \to B$?

Solution We received no correct solutions. The answer to the question is “no”. Let A be the finite commutative unital ring $\mathbb{F}_2[x,y]/(x^2,y^2)$. Let B be a finite commutative unital ring in which every ideal is principal, and let $f\colon A \to B$ be a ring homomorphism. It now suffices to show that f cannot be injective.
First note that the ring homomorphism f forces $2 = 0$ in B. By assumption there exists an element $z \in B$ such that we have the identity $\big(f(x), f(y)\big) = (z)$ as ideals of B. Hence there exist $b, c \in B$ such that $f(x) = bz$ and $f(y) = cz$, so $f(xy) = bcz^2$. Writing $z = df(x) + ef(y)$ with $d, e \in B$, we get $z^2 = d^2 f(x^2) + 2def(xy) + e^2 f(y^2) = 0$. We deduce that $f(xy) = 0$, hence f is not injective, as desired.

On Problem 2013-1/A. We would like to thank Jan Stevens for pointing out that Problem 2013-1/A in fact originated from his work (see [2]), and that there is a connection with Coxeter’s *frieze patterns* (see [3, 4]).

References

1. B. Kostant and N. Wallach, Gelfand–Zeitlin theory from the perspective of classical mechanics, Studies in Lie theory, *Progress in Mathematics* 243, 319–364, http://arxiv.org/abs/math/0408342v2.
2. J. Stevens, On the versal deformation of cyclic quotient singularities, *Singularity theory and its applications, Part I*, Lecture Notes in Math. 1462, Springer, Berlin, 1988/1989, 302–319.
3. J. H. Conway and H. S. M. Coxeter, Triangulated polygons and frieze patterns, *Math. Gaz.* 57 (1973), 87–94 and 175–183.
4. H. S. M. Coxeter, Frieze patterns, *Acta Arith.* 18 (1971), 297–310.