

# Problemen

| Problem Section

This Problem Section is open to everyone; everybody is encouraged to send in solutions and propose problems. Group contributions are welcome.

For each problem, the most elegant correct solution will be rewarded with a book token worth €20. At times there will be a Star Problem, to which the proposer does not know any solution. For the first correct solution sent in within one year there is a prize of €100.

When proposing a problem, please either include a complete solution or indicate that it is intended as a Star Problem. Electronic submissions of problems and solutions are preferred (problems@nieuwarchief.nl).

The deadline for solutions to the problems in this edition is 1 March 2014.

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**Problem A** (folklore, communicated by Jaap Top)

Does there exist an integer  $n > 1$  such that the set of leading digits of  $2^n, 3^n, \dots, 9^n$  is equal to  $\{2, 3, \dots, 9\}$ ?

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**Problem B** (proposed by Bart de Smit and Hendrik Lenstra)

Rings are unital, and morphisms of rings send 1 to 1.

Let  $A$  and  $B$  be commutative rings. Suppose that there exists a ring  $C$  such that there are injective morphisms  $A \rightarrow C$  and  $B \rightarrow C$  of rings. Show that there exists a commutative such ring.

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**Problem C** (proposed by Jinbi Jin)

Let  $C(\mathbb{R}, \mathbb{R})$  denote the set of continuous maps from  $\mathbb{R}$  to itself. A (not necessarily continuous) map  $f: C(\mathbb{R}, \mathbb{R}) \rightarrow C(\mathbb{R}, \mathbb{R})$  is called *good* if it satisfies, for all  $s, t \in C(\mathbb{R}, \mathbb{R})$ , the identity

$$f(s \circ t) = f(s)f(t),$$

where the product on the right hand side is the point-wise multiplication of maps.

- Find a non-constant good map  $f: C(\mathbb{R}, \mathbb{R}) \rightarrow C(\mathbb{R}, \mathbb{R})$ .
- Show that  $f(\exp) = 0$  for all non-constant good maps  $f: C(\mathbb{R}, \mathbb{R}) \rightarrow C(\mathbb{R}, \mathbb{R})$ . (Here,  $\exp$  is given by  $x \mapsto e^x$ .)

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**Edition 2013-2** We received solutions from Leon van den Broek (Nijmegen), Alex Heinis (Amsterdam), Jos van Kan (Delft), Thijmen Krebs (Nootdorp), Javier Sánchez-Reyes (Castilla-La Mancha, Spain) and Ángel Plaza (Las Palmas de Gran Canaria, Spain), and Robert van der Waall (Huizen).

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**Problem 2013-2A** (based on a problem proposed by Gerard Renardel de Lavalette)

We have two hourglasses,  $A$  for  $a$  seconds and  $B$  for  $b$  seconds, where  $a$  and  $b$  are relatively prime integers and  $0 < a < b$ . Let  $t_0$  be an integer with  $t_0 \geq b + (\frac{1}{2}a - 1)^2$ . Show that  $A$  and  $B$  can be used to identify the time  $t = t_0$  if the upper bulbs are empty at  $t = 0$ .

**Remark.** The original problem received from the proposer was to prove a slightly stronger result. Let  $m$  be the remainder of  $b$  upon division by  $a$ . The original problem was to prove that for any integer  $t_0 > b + m(a - m) - a$ , the time  $t = t_0$  can be identified using  $A$  and  $B$ .

**Solution** We received only one correct solution, from Thijmen Krebs, who will receive the book token. The following solution is based on that solution.

Let  $m$  be the remainder of  $b$  upon division by  $a$ . For any integer  $T$  that is a multiple of  $a$  or  $b$ , we can use the following strategy:

- while  $t < T$ , turn each hourglass whenever it is empty;
- while  $t \geq T$ , turn *both* hourglasses whenever at least one is empty.

If we apply this strategy to  $T = b$ , then we turn both hourglasses at the times  $t = b + km$  for  $k = 0, 1, 2, \dots$

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If we apply this strategy to  $T = a(1 + \frac{b-m}{a}) = b + (a - m)$ , then we turn both hourglasses at the times  $t = b + k(a - m)$  for  $k = 1, 2, 3, \dots$

In particular, all elements of the following set are measurable times:

$$S = \{b + km : 0 \leq k < a - m\} \cup \{b + k(a - m) : 0 < k \leq m\}.$$

As  $a$  and  $b$  are coprime, so are  $m$  and  $a$ , hence  $S$  contains an element of each residue class modulo  $a$ . Moreover, the maximal element of  $S$  is  $b + m(a - m)$ .

Before starting the strategy above, we can measure any non-negative integer multiple of  $a$  seconds using  $A$ , while letting  $B$  stay empty. In particular, we can measure any time  $t_0 \geq b + m(a - m) - a + 1$ .

Finally, note  $m(a - m) \leq (\frac{1}{2}a)^2$ , so  $b + m(a - m) - a + 1 \leq b + (\frac{1}{2}a)^2 - a + 1 = b + (\frac{1}{2}a - 1)^2$  and we can measure any time  $t_0 \geq b + (\frac{1}{2}a - 1)^2$ .

**Problem 2013-2B** (folklore, communicated by Jeanine Daems)

In a two-player game, players take turns drawing a number of coins from a pile that starts with  $n$  coins. The first player takes at least one coin from the pile, but not all. In the subsequent turns, each player takes at least one coin, and at most twice the number of coins taken in the previous turn. The player who takes the last coin wins. For which numbers  $n$  can the first player win?

**Solution** We received correct solutions from Alex Heinis and Thijmen Krebs. The book token is awarded to Alex Heinis. The game is known as *Fibonacci Nim*, and the first player can win for those integers  $n > 1$  that are not a Fibonacci number.

Let  $(F_k)_{k \geq 1}$  be the Fibonacci sequence:  $F_1 = 1, F_2 = 2$  and  $F_{k+2} = F_{k+1} + F_k$  for  $k \geq 1$ . The proof uses *Zeckendorf's theorem*: every positive integer can uniquely be written as the sum of non-consecutive Fibonacci numbers. Let  $z$  be the function on the positive integers that assign to  $m$  the smallest Fibonacci number occurring in the Zeckendorf decomposition of  $m$ . E.g., we can write  $20 = 13 + 5 + 2 = F_6 + F_4 + F_2$  and  $z(20) = F_2 = 2$ .

We define a position in this game to be a pair  $(m, d)$  where  $m$  is number of coins left on the pile and  $d$  the maximal number of coins that may be taken (by the player who is to move). The initial position is  $(n, n - 1)$  and the final positions are those of the form  $(0, d)$ . Call a position  $(m, d)$  'good' if it is non-final and  $d \geq z(m)$ ; call it 'bad' otherwise.

**Lemma.** *Let  $(m, d)$  be a good position. There exists a move to a bad position.*

*Proof.* Write  $m = F_{k_1} + \dots + F_{k_r}$  for the Zeckendorf decomposition, with  $k_i \geq k_{i+1} + 2$ . By assumption,  $d$  is at least  $F_{k_r}$ . Our move is to take exactly  $F_{k_r}$  coins. The new position is  $(m - F_{k_r}, 2F_{k_r})$ . This is a bad position: in the case  $r = 1$  it is even final, and otherwise it follows from  $2F_{k_r} < F_{k_r-1}$ . □

**Lemma.** *Let  $(m, d)$  be a non-final bad position. All moves lead to a good position.*

*Proof.* Write  $F_k = z(m)$ . By assumption we have  $d < F_k$ . Suppose we take  $x$  coins, for some  $x \in \{1, \dots, d\}$ . Let  $t \geq 0$  be the even number such that  $F_{k-t-2} \leq x < F_{k-t}$ . Then

$$F_k - F_{k-t} < F_k - x \leq F_k - F_{k-t-2}$$

hence

$$F_{k-1} + F_{k-3} + \dots + F_{k-t+1} < F_k - x \leq F_{k-1} + F_{k-3} + \dots + F_{k-t-1},$$

so  $z(F_k - x) \leq F_{k-t-1}$ , which is smaller than  $2F_{k-t-2} \leq 2x$ . Note further that  $z(F_k - x) = z(m - x)$ . Hence  $(m - x, 2x)$  is a good position. □

Together the lemmas show that the good positions are exactly the winning ones. The initial position  $(n, n - 1)$  is good if and only if  $z(n) \leq n - 1$ , i.e., if and only if  $n$  is not Fibonacci.

# Oplossingen

| Solutions

**Problem 2013-2C** (proposed by Bas Edixhoven and Maarten Derickx)

Let  $ABCD$  be a convex quadrilateral inside a plane  $U$  in  $\mathbb{R}^3$ . Suppose that  $ABCD$  is not a parallelogram. Show that there exist a plane  $V$  in  $\mathbb{R}^3$  and a point  $P \in \mathbb{R}^3 - (U \cup V)$  such that if a light source is placed in  $P$ , then the shadow of  $ABCD$  on  $V$  is a square.

**Solution** We received solutions from Leon van den Broek, Alex Heinis, Jos van Kan, Javier Sánchez-Reyes and Ángel Plaza, and Robert van der Waall. The book token goes to Jos van Kan. The main idea of the solution is to pick the plane  $V$  and the point  $P$  in such a way that the projection on  $V$  of the intersection of the lines  $AB$  and  $CD$ , and that of  $BC$  and  $AD$  is ‘at infinity’. Some extra conditions on  $P$  related to the diagonals and consecutive edges will then ensure that the projection of  $ABCD$  on  $V$  is a square.

If the lines  $AB$  and  $CD$  intersect, denote their intersection by  $X_1$ . Similarly, if  $BC$  and  $AD$  intersect, denote their intersection by  $X_2$ .

We consider three cases, the first of which is the following.

**Case 1.** The lines  $AB$  and  $CD$  intersect, and so do  $BC$  and  $AD$ . Moreover, both of  $AC$  and  $BD$  intersect the line  $X_1X_2$ .

First note that  $X_1X_2$  does not intersect the quadrilateral  $ABCD$ , as  $ABCD$  is convex. Let  $Y_1$  be the intersection of  $AC$  and  $X_1X_2$ , and likewise, let  $Y_2$  be the intersection of  $BD$  and  $X_1X_2$ . Let  $W$  be a plane that has as intersection the line  $X_1X_2$  with  $U$ . In particular,  $U \neq W$ . Let  $\Gamma_1, \Gamma_2$  be the circles in  $W$  with the segments  $\overline{X_1X_2}, \overline{Y_1Y_2}$  as diameter, respectively. Let  $P$  be an intersection of  $\Gamma_1$  and  $\Gamma_2$ , and let  $V$  be any plane parallel to  $W$  such that the quadrilateral  $ABCD$  lies between  $V$  and  $W$ . This intersection exists as one of  $Y_1, Y_2$  lies between  $X_1$  and  $X_2$ , and the other does not.

Then note that  $P$  does not lie in  $U$ , as the points  $X_1, X_2, Y_1, Y_2$  are pairwise distinct, and that  $P$  does not lie in  $V$ , as  $P$  lies in  $W$ , which is parallel to  $V$ . Hence  $P \in \mathbb{R}^3 - (U \cup V)$ .

Now let  $A_0, B_0, C_0, D_0$  be the respective intersections of  $AP, BP, CP, DP$  with  $V$ . They exist, as the given lines intersect in  $P$  with  $W$ , which is parallel to  $V$ . By construction of  $V$ , and as  $X_1X_2$  does not intersect the quadrilateral  $ABCD$ , it now suffices to show that  $A_0B_0C_0D_0$  is a square in  $V$ .

Let  $l$  be a line in  $U$ , not equal to  $X_1X_2$ . Write  $\mathcal{P}(l)$  for the unique plane through  $l$  and  $P$ , write  $\mathcal{I}(l)$  for the intersection line of  $W$  with  $\mathcal{P}(l)$ , and write  $\mathcal{I}_0(l)$  for the intersection line of  $V$  with  $\mathcal{P}(l)$ . (So for example,  $\mathcal{I}_0(AB) = A_0B_0$ .) As  $V$  and  $W$  are parallel, it follows that for all lines  $l, m$  in  $U$ , the angle between  $\mathcal{I}(l)$  and  $\mathcal{I}(m)$  is equal to the one between  $\mathcal{I}_0(l)$  and  $\mathcal{I}_0(m)$ . Note that a square is a quadrilateral such that

- every two successive edges are perpendicular, and
- the diagonals are perpendicular.

Therefore, to show that  $A_0B_0C_0D_0$  is a square, it suffices to show that the  $\mathcal{I}(e)$  (with  $e$  an edge or a diagonal of the quadrilateral  $ABCD$ ) satisfy the above properties.

Now we simply note that

$$\mathcal{I}(AB) = \mathcal{I}(CD) = PX_1, \quad \mathcal{I}(BC) = \mathcal{I}(AD) = PX_2,$$

and that

$$\mathcal{I}(AC) = PY_1, \quad \mathcal{I}(BD) = PY_2,$$

so by construction of  $P$ , the quadrilateral  $A_0B_0C_0D_0$  is a square, as desired.

For the remaining cases, we will only state them, and the corresponding construction of  $P$ , as the proof (and the construction of  $V$ ) is done in the same way.

**Case 2.** The lines  $AB$  and  $CD$  intersect, and so do  $BC$  and  $AD$ . Moreover, at most one of  $AC$  and  $BD$  intersects the line  $X_1X_2$ .

Note here that at least one of  $AC$  and  $BD$  intersects the line  $X_1X_2$ , as  $AC$  and  $BD$  intersect, so exactly one of them intersects  $X_1X_2$ . We assume without loss of generality that  $AC$  and

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$X_1X_2$  intersect, and let  $Y$  be their intersection. Note that  $Y$  lies between  $X_1$  and  $X_2$ , as the line  $AC$  intersects the segment  $\overline{BD}$ , which is parallel to  $X_1X_2$ . Let  $W$  be any plane that has as intersection the line  $X_1X_2$  with  $U$ , and let  $\Gamma$  be the circle in  $W$  with diameter  $\overline{X_1X_2}$ . Then we take  $P$  to be an intersection of  $\Gamma$  with the line through  $Y$  perpendicular to  $X_1X_2$ .

**Case 3.** Exactly one of the pairs of lines  $(AB, CD)$  and  $(BC, AD)$  intersect.

We assume without loss of generality that  $AB$  and  $CD$  intersect, and let  $X$  denote this intersection. Let  $l$  be the line through  $X$  parallel to  $BC$  (hence also to  $AD$ ). Then the lines  $AC$  and  $BD$  both intersect  $l$ , as they intersect  $BC$ . Let  $Y_1$  and  $Y_2$  be their respective intersections. Then  $X$  lies between  $Y_1$  and  $Y_2$ , as for  $S$  the intersection of  $AC$  and  $BD$ , the line  $XS$  intersects the segments  $\overline{BC}$  and  $\overline{AD}$ , which are parallel to  $l$ . Let  $W$  be any plane that has  $l$  as intersection with  $U$ , and let  $\Gamma$  be the circle with diameter  $\overline{Y_1Y_2}$ . Then we take  $P$  to be an intersection of  $\Gamma$  with the line through  $X$  perpendicular to  $l$ .

**References.** This problem turned out to be rather well-known, as we received a lot of references. Thanks to Leon van den Broek, Javier Sánchez-Reyes and Ángel Plaza, and Robert van der Waall for these. The references given were, respectively,

- L. van den Broek, Welke schaduwbeelden, *Euclides* 64, nr. 3 (1988, in Dutch).
- Problem 72 of H. Dörrie, *100 great problems of elementary mathematics, their history and solution* (translation of Triumph der Mathematik, 1932), reworked in 2010 by M. Woltermann <http://www2.washjeff.edu/users/MWoltermann/Dorrie/72.pdf>.
- E.J. Hopkins and J.S. Hails, *An Introduction to Plane Projective Geometry*, Clarendon Press (1953).

