Problemen

| Problem Section

This Problem Section is open to everyone; everybody is encouraged to send in solutions and propose problems. Group contributions are welcome.
For each problem, the most elegant correct solution will be rewarded with a book token worth 20 Euro. At times there will be a Star Problem, to which the proposer does not know any solution. For the first correct solution sent in within one year there is a prize of 100 Euro.
When proposing a problem, please either include a complete solution or indicate that it is intended as a Star Problem. Electronic submissions of problems and solutions are preferred (problems@nieuwarchief.nl).
The deadline for solutions to the problems in this edition is 1 March 2013.

Problem A (proposed by Jan Turk)

Let $\phi(n)$ denote the Euler totient function. Find the set of limit points of the sequence $\big(\phi(n)/n\big)_{n=1}^{\infty}$.

Problem B

Find nonzero integers c_0, c_1, c_2, c_3 such that the sequence given by $a_1 = 1$, $a_2 = 12$, $a_3 = 68$, $a_4 = 504$, and

$$a_{n+4} = c_0 a_n + c_1 a_{n+1} + c_2 a_{n+2} + c_3 a_{n+3} \quad (n > 0)$$

consists of positive terms and has the property that a_m divides a_n whenever m divides n.

Problem C (proposed by Johannas Winterink)

A circle in $\mathbb{R}^2$ is called *Apollonian* if its centre coordinates and radius are all integers.
Do there exist eleven distinct Apollonian circles $A, B, C, T_1, \ldots, T_8$ such that for $i = 1, \ldots, 8$, the circle T_i is tangent to A, B, and C?

Edition 2012-2 We received solutions from Wouter Cames van Batenburg (Leiden), Cor Hurkens (Eindhoven), Thijmen Krebs (Nootdorp), José H. Nieto (Maracaibo) and Hans Zantema (Eindhoven).

Problem 2012-2/A Let P and Q be distinct points in the plane. Let $n \geq 2$. Assume n distinct lines through P but not through Q are given, as well as n distinct lines through Q but not through P. Let T be a collection of $2n$ intersection points of these lines. Suppose that the (unoriented) angle between the lines RP and RQ is the same for all R in T, and not a multiple of $\frac{1}{4}\pi$. Show that T can be partitioned into subsets of at least three elements each, such that every subset consists of the vertices of a regular polygon.

Rectification. The common angle in this problem should not be a multiple of $\pi/4$. (Thanks to Thijmen Krebs for pointing this out.)

Solution We received a correct solution from Thijmen Krebs.
All angles are oriented angles modulo π, unless stated otherwise. Let α be the unoriented angle modulo π of the common angle of the $\angle PRQ$, where $R \in T$.

Observation 1. *Every line through P (resp. Q) contains exactly two points of T.*

Proof. Let L be a line through P. As Q is not on this line, there is a unique isosceles triangle with base inside L, top Q, and base angles α. Hence there are at most two points of T on any given line through P. But since we have n lines going through P, and $\#T = 2n$, it must follow that every line must contain exactly two points of T. The same argument holds for Q. □

Observation 2. *The set T is a subset of the union of two distinct circles intersecting at P and Q.*

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Proof. Note that by the inscribed angle theorem, the subset T_+ of T consisting of the points $R \in T$ such that $\angle PRQ = \alpha$ lies on a circle Γ_+ containing P and Q, and that the subset T_- of T consisting of the points $R \in T$ such that $\angle PRQ = -\alpha$ also lies on a circle Γ_- containing P and Q. Moreover, these circles are distinct since $\alpha \neq \frac{1}{2}\pi$ by assumption. □

We now define two maps $f_P, f_Q : T_+ \to T_-$ as follows. Let $R \in T_+$. Then $f_P(R)$ (resp. $f_Q(R)$) is the unique intersection point of the line RP (resp. RQ) with Γ_- not equal to P (resp. Q). This map is well-defined, as for $R \in T_+$, we have $\angle P f_P(R) Q = \angle P f_Q(R) Q = -\alpha$, hence $f_P(R), f_Q(R) \in T_-$ by Observation 1.

Observation 3. *The maps f_P and f_Q are bijections. In particular, $\#T_+ = \#T_- = n$.*

Proof. We simply note that the inverse is given by sending $R \in T_-$ to the unique intersection point of the line RP (resp. RQ) with Γ_+ not equal to P (resp. Q). □

Observation 4. *The maps $f_P^{-1} f_Q$ and $f_Q f_P^{-1}$ are rotations by 4α (as an oriented angle modulo 2π) on T_+ and T_-, respectively (with centres those of Γ_+ and Γ_-, respectively).*

Proof. Let $R \in T_+$. Then $\angle PRQ = \angle Q f_Q(R) P = \alpha$, it follows that $\angle RP f_P^{-1} f_Q(R) = \angle RP f_Q(R) = 2\alpha$. Hence if C_+ is the centre of Γ_+, then $\angle RC_+ f_P^{-1} f_Q(R) = 4\alpha$, as an oriented angle modulo 2π. The same argument works for $f_Q f_P^{-1}$. □

Now we note that the orbits of T_+ (resp. T_-) under the action of $f_P^{-1} f_Q$ (resp. $f_Q f_P^{-1}$) all have the same length by the above, which hence divides n, so it follows that $f_P^{-1} f_Q$ and $f_Q f_P^{-1}$ have order dividing n. Hence $4n\alpha = 0$ modulo 2π, so $\alpha = 0$ modulo $\pi/2n$. As we assumed that α was not a multiple of $\frac{1}{4}\pi$, it follows that orbits of length at most 2 cannot occur. Orbits of higher length are sets whose vertices form a regular polygon with at least three vertices, so we are done.

Problem 2012-2/B Show that there exist an $n \geq 1$, a polynomial $P \in \mathbb{Z}[X, Y_1, \ldots, Y_n]$ and an infinite set S of positive integers such that the set

$$\left\{(y_1, \ldots, y_n) \in \mathbb{Z}^n : P(k, y_1, \ldots, y_n) = 0\right\}$$

is empty for all $k < 0$ and has precisely k elements for all $k \in S$.

Solution We received a correct solution from Thijmen Krebs.
An example can be deduced from Jacobi's four-square theorem. It states that for each positive integer p, the number of solutions $(y_1, y_2, y_3, y_4) \in \mathbb{Z}^4$ to

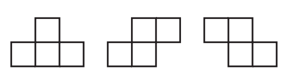
$$y_1^2 + y_2^2 + y_3^2 + y_4^2 = p$$

is $r_4(p) = 8 \sum_{d \in D} d$, where D is the set of divisors of p that are not multiples of 4. In particular, if p is prime we have $r_4(p) = 8(p+1)$.
Set $n = 4$ and let $P \in \mathbb{Z}[X, Y_1, Y_2, Y_3, Y_4]$ be the polynomial

$$P = 8(Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 + 1) - X.$$

Define $S = \{8(p+1) : p \text{ prime}\}$. The equation $P(k, y_1, y_2, y_3, y_4) = 0$ has no solutions for $k < 0$. For $k = 8(p+1) \in S$ the equation reduces to $y_1^2 + y_2^2 + y_3^2 + y_4^2 = p$, which has $r_4(p) = k$ solutions.

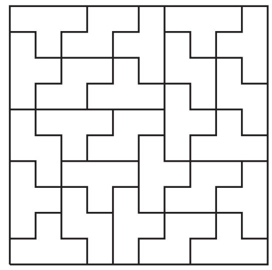
Problem 2012-2/C Is it possible to tile a 30 by 30 square grid using the following blocks?



Solution We received correct solutions from Wouter Cames van Batenburg, Cor Hurkens, Thijmen Krebs, José H. Nieto and Hans Zantema. The book token goes to José H. Nieto.

Oplossingen | Solutions

There exists a tiling as desired. In fact, we can already tile a 10×10 grid.



Note that we do not even need both types of Z-tiles.
More generally, an $n \times m$ grid can be tiled with the given pieces if and only if n and m are at least 4, nm is divisible by 4, and (n, m) is not $(6, 6)$, $(6, 10)$ or $(10, 6)$.