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Poincaré and Analysis Situs, the beginning of algebraic topology

In 1895 Henri Poincaré published his topological work 'Analysis Situs'. A new subdiscipline in mathematics was born. Analysis Situs was an inspiration to new fields like algebraic topology, Morse theory and cobordism. With use of today's knowledge and notation, Dirk Siersma views back to this historical work of Poincaré.

What was the impact of Poincaré on topology? He introduced the concept of manifold in any dimension and defined homologies and fundamental groups. This was the starting point for the development of algebraic topology. Although he discusses the general case, his work is quite concrete. He works often with examples and makes computations. This was his way to get intuition. His topological work 'Analysis Situs' [5] appeared in 1895. Before, in 1892 he had published a short (four pages) announcement in *Comptes Rendus* [4].

Analysis Situs describes the relative position between objects (points, lines, surfaces) without bothering about their sizes.

Analysis Situs is written in an intuitive style, which is quite different from the present mathematical writing. It reads sometimes like a novel. It is divided into 18 short chapters and consists of 121 pages. Definitions and theorems are not so often mentioned as such. Poincaré is not always precise and at some places there are gaps and mistakes. Due to criticism of other researchers (e.g. Heegard) he responded by adding supplements (all together five) during the period 1899–1904. In the last (fifth) supplement he stated correctly his question, which we call now the Poincaré conjecture (which was proved by Perelman in 2003).

Analysis Situs and the supplements contain (in a preliminary stage) many seeds for further developments: algebraic topology, Morse theory, topology of algebraic varieties and cobordism. This article is not a historical survey of Poincaré's topological works. It reports on my experiences while reading in his work. At several places I will be anachronistic and use some of today's knowledge and notations and view back to Poincaré's work.

Several books have been written about Poincaré's topological work. We mention first John Stillwell's English translation [6] of Analysis Situs and the five supplements, which appeared under the title *Papers on Topology* [6]. As further reading I propose the article of Sakaria [7] and the book of Scholz [8].

Why Analysis Situs?

How did Poincaré come to study analysis situs? Most of his work was of a geometric nature: differential equations (in his dissertation), dynamical systems and the theory of automorphic functions. This last subject is related to non-Euclidean geometry. In his study of differential equations he was also looking to more qualitative aspects. e.g. the indices of zero's of a vector field and more global aspects of the theory. An example is the index formula for vector fields: the sum of the indices on a surface of genus p is equal to $2-2p$. So it only depends on the 'shape' of the surface. Moreover he wanted to generalize this to higher dimensions. He saw a need for extending the concept of connectivity (in the surface case related to the genus) to higher dimensions.

He also looked at spaces of differential equations on algebraic curves. Depending on genus and branching order he constructed an object, depending on many coordinates, which he called *multiplicité*. In his theory of automorphic functions he found in a similar way a *multiplicité* of Fuchsian groups with fundamental region a surface of genus p and given branching. Also in his work on double integrals on $\mathbb{C}^2$ he entered the theory of submanifolds of $\mathbb{R}^4$. At several places he talks about the need of a 'hypergeometrical language'. In 1892 [4] it was so far that he announced Analysis Situs as new subdiscipline in mathematics.

Manifolds

Before Poincaré the concept of (smooth) manifold was already used in the two dimensional case: classification of embedded surfaces in $\mathbb{R}^3$ was carried out by Möbius in 1863. There was also a description by identification and by fundamental region. The notion of an n-dimensional manifold was already around and used by e.g. Betti.

Poincaré does not give an abstract definition of a manifold, but describes them by constructions. See also Figure 1.

The first construction was by a set of p *equations* in $\mathbb{R}^{n+p}$ with Jacobian matrix of maximal rank together with some inequalities. This is nowadays called the submersion condition.

With the second construction he could describe more complicated situations: by *local parametrizations*; in modern language a local embedding $\mathbb{R}^m \to \mathbb{R}^n$. Poincaré relates the first construction to the second by the implicit function theorem. He also discusses the overlap between several local parametrizations, as a chain (like in the case of analytic continuation of complex functions) but without the concept of atlas.

In chapter 10 he considered a third construction *Geometric Representation*, where a certain number (one or more) of polyhedra in ordinary space are glued together by identifying pairs of faces. (Of course the gluing has to be done in such a way that the result is a manifold!)

Main examples are cube manifolds, which we will discuss later. Anyhow in the geometric representation Poincaré made most of his computations.

The definitions also allowed manifolds with boundaries, e.g. the solid torus, the n-ball and the regions between two spheres.

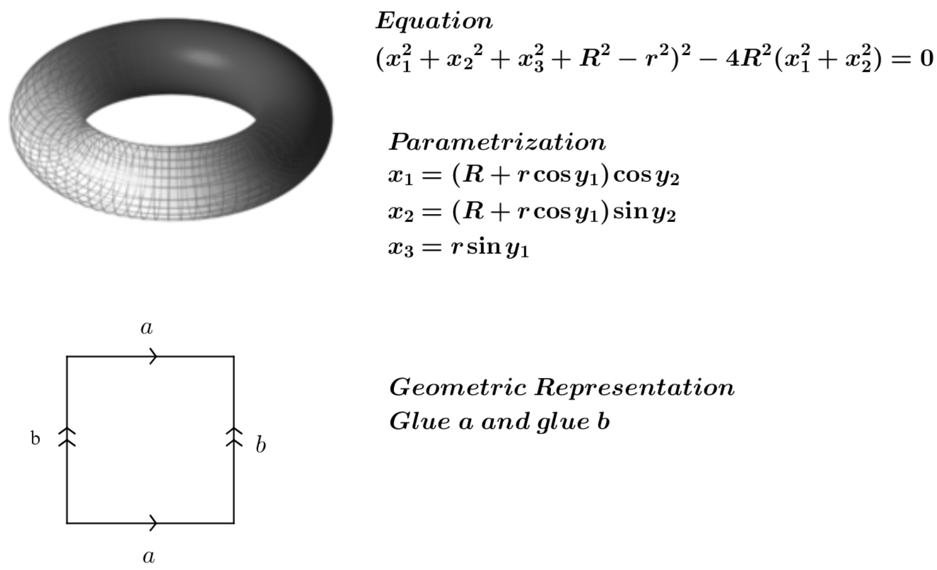

Figure 1 Three types of definition of a manifold

Homologies and Betti numbers

Poincaré wanted to study the (higher) connectedness of a manifold. For this he introduced a calculus with submanifolds. He wrote:

$$k_1V_1 + k_2V_2 \sim k_3V_3 + k_4V_4, \quad k_i \in \mathbb{N},$$

when there exists a submanifold W with a boundary, which is composed of k_i copies of closed submanifolds V_i (Figure 2 and 3). He said: "Relations of these forms are called homologies." And moreover: "Homologies can be combined like ordinary equations." He defined the submanifolds $V_1, \ldots, V_\lambda$ to be independent if they are not connected by any homology with integral coefficients. He defined the connectivity of V with respect to manifolds of dimension m as P_m, if there exist $P_m - 1$ closed submanifolds of dimension m, which are linearly independent, but not less. So we get a set of numbers $P_1, \ldots, P_n$ for each manifold V of dimension n. He called this the sequence of *Betti numbers*. Note that these Betti numbers are 1 higher than today's Betti numbers (which are the ranks of homology groups). The Betti numbers occur also in the last chapter, where he generalized the Euler formula for surfaces to manifolds.

To allow negative coefficients he used the concept of orientation (Klein, van Dyck) in relation with the sign of Jacobian determinant of transition maps in the second construction of manifolds. This allowed him to write:

$$k_1V_1 + k_2V_2 + \cdots + k_\lambda V_\lambda \sim 0, \quad k_i \in \mathbb{Z}.$$

Although this is a linear combination of submanifolds, Poincaré did not consider the group theoretic aspects. He exploited the idea of Betti to consider 'taking-the-boundary' in order to measure connectivity. This became the main tool in geometric homology theories and cobordism.

In Analysis Situs he did not consider torsion. Poincaré allowed divisions: $4v_1 \sim 0$ implies $v_1 \sim 0$. In modern language he worked only with the free part of the homology groups. He discussed torsion in the first supplement (after criticism of Heegaard).

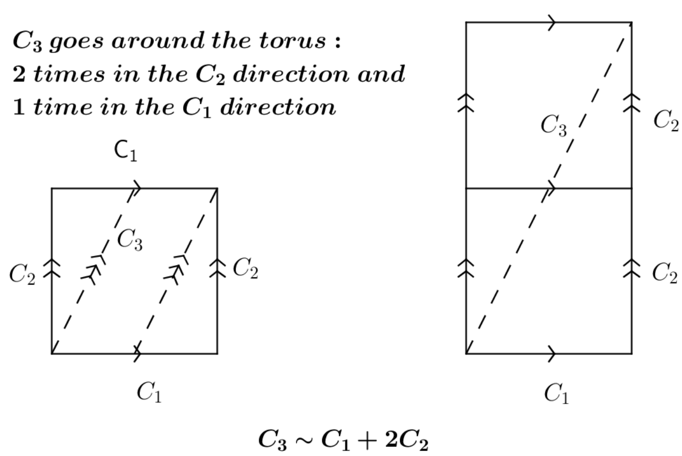

Figure 2 Homology on the torus

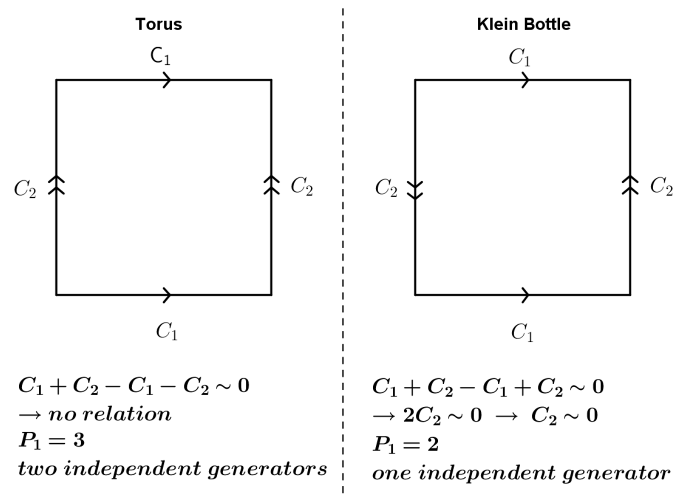

Figure 3 Betti numbers and homologies

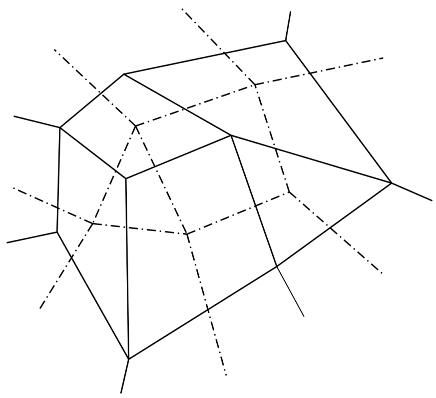

Figure 4 Cell decomposition and its dual

Poincaré duality

In examples it turned out that Betti numbers were symmetric around $\frac{n}{2}$. This is the so-called Poincaré duality, which is valid for oriented closed manifolds. He gave in chapter 9 a sketch of the proof of $P_k = P_{n-k}$. The central idea is to consider the intersection number of two (transversal) submanifolds of complementary dimension. For each intersection point this is the local intersection number $+1$ or -1, according to orientation of a system of tangent vectors. He defined the global intersection $N(V, V')$ as the sum of the local intersection numbers. He also claimed the independence under homology relations. It follows that for every $n-k$ cycle C there exists a closed k-dimensional submanifold V such that $N(V, C) \neq 0$. This explained the duality (anyhow for the free part of homology).

The criticism of Heegard (who showed him a counter example) was reason for him to describe in more detail the difference between homology with division and homology without division (including torsion). This was also a reason to produce a new proof for the duality (in the first supplement), where he looked to a decomposition into cells (homeomorphic to the ball) together with a dual cell decomposition (Figure 4).

The fundamental group

In chapter 12 Poincaré introduced the fundamental group. He knew from the theory of Fuchsian groups already the relation between closed curves on a surface and the substitutions in a system of multivalued functions. In the case of a 2-torus one can e.g. consider the two angular coordinates (which are defined local). See Figure 5. In fact if ϕ is such a coordinate its differential $d\phi$ is well defined and it gives rise to a multivalued function on the torus. The integral of $d\phi$ over a closed contour gives a integer multiple of 2π. The use of substitutions is quite typical for the period 1880–1920. It occurred also in systems of solutions of differential equations, following these solutions around different loops around singularities.

In general a contour produces a substitution in a multivalued function and a composition of contours results in a composition of substitutions. Multivalued functions can be interpreted as univalued on a certain covering space of the manifold. Substitutions act as deck transformations. In fact the 'group of substitutions' is a holomorphic image of the fundamental group. Be aware that no abstract concept of group was known. A 'group' was always connected with an action.

Poincaré's composition of closed curves (contours) with common base point is not commutative, but he used an additive notation. A first definition of equivalent contours (written as $\equiv$) was close to the homology relation. This definition was not completely clear and some corollaries were incorrect. He comes back to it in the fifth supplement, where he used continuous homotopy between contours in the modern sense of the term.

He also made the difference clear between homology ($\sim$) and homotopy ($\equiv$). In homotopy:

- composition is not commutative
- all contours have the same base point
- $nA \equiv 0$ not necessarily implies $A \equiv 0$ (note that Poincaré did not consider torsion in homology in 1895)

A next step (in chapter 13) was to describe the fundamental group by generators and relations. Generators are a finite number of principal substitutions $S_1, \ldots, S_p$ that correspond to closed contours $C_1, \ldots, C_p$ such that any other contour is equivalent to a combination of these fundamental contours in a certain order. These fundamental contours are not, in general, independent, and there are certain relations between them which are called *fundamental equivalences*. The fundamental equivalences enable us to know the structure of the group.

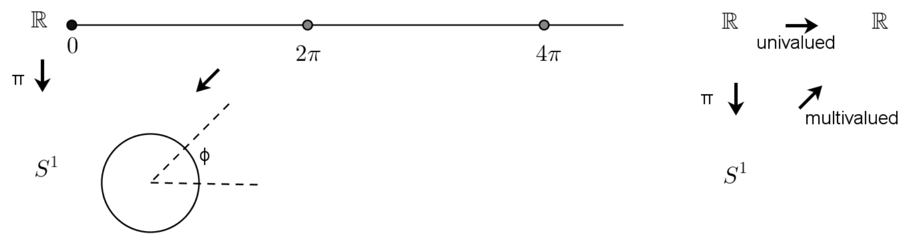

Figure 5 Angular coordinate as a multivalued function

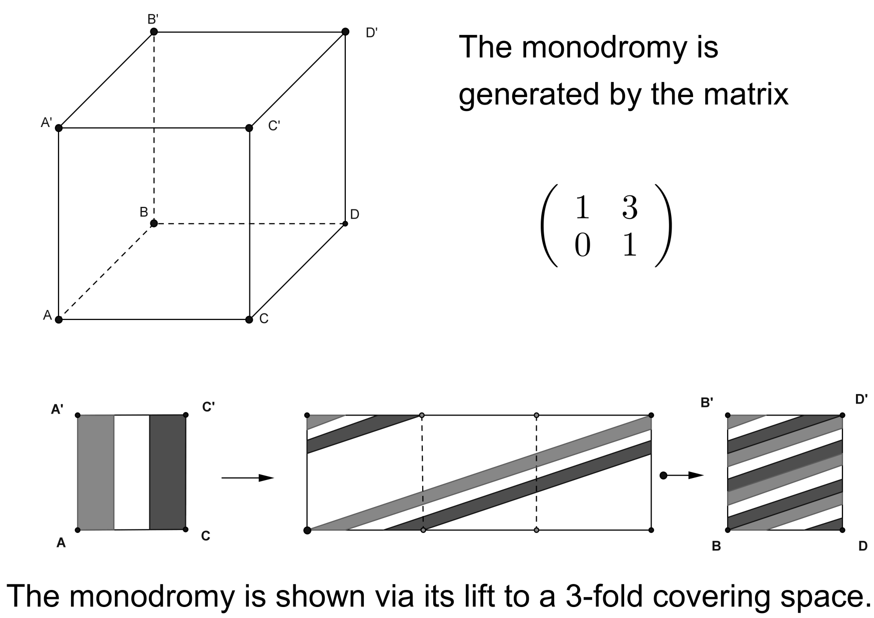

Figure 6 The construction of a cube manifold

Examples: The cube manifolds

Poincaré studied the cube manifolds as an important set of three-dimensional examples.

Consider the manifold V as orbit space of a group generated by the following three transformations:

$$g_i : \mathbb{R}^3 \to \mathbb{R}^3 \quad (i = 1, 2, 3)$$

defined by

$$g_1(x, y, z) = (x + 1, y, z),$$

$$g_2(x, y, z) = (x, y + 1, z),$$

$$g_3(x, y, z) = (ax + by, cx + dy, z + 1),$$

where a, b, c, d are integers and $ad - bc = 1$.

The fundamental domain is a unit cube. One identifies opposite faces by the following maps: the identity for the x and y coordinates and in the z direction a diffeomorphism, which is generated by a linear map (Figure 6).

Vertical sections of the cube correspond to tori. We can consider the cube manifold as a torus bundle over a circle. Such a bundle has a so-called *monodromy*. Cut the circle and look to the induced (trivial) bundle over the interval. The monodromy is the gluing map of the tori above the two end points. For cube manifolds the monodromy is generated by the linear map.

Figure 7 A handle body of genus 2

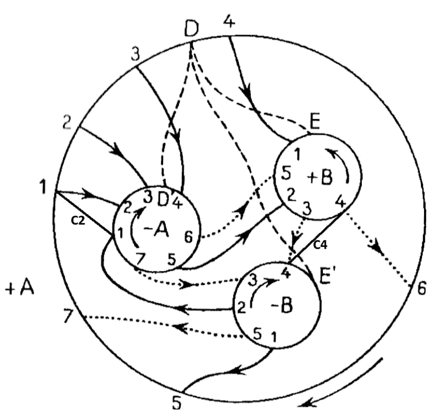

Figure 8 The Poincaré sphere

Poincaré sphere: explicit computation

Start with a 3-ball. On its boundary S^2 one chooses two pairs of discs $(+A, -A)$ and $(+B, -B)$. Glue $+A$ (which is shown as the outside region in Figure 8) with $-A$ and glue $+B$ with $-B$. Call the new boundary contours δA and δB. The resulting 3-ball with two handles is the handle body V_1. Its boundary W is a surface of genus 2.

Consider the contours:

$$C_1 = \delta A,\ C_2 = \text{[connects +A and −A]},\ C_3 = \delta B,\ C_4 = \text{[connects +B and −B]}.$$

These four cycles are the fundamental 1-cycles of W, the fundamental group is free and abelian and equal to the first homology group of W. The fundamental group of V_1 is also free and abelian, generated by C_2 and C_4 and equal to the first homology group of V_1 (C_1 and C_3 are principal cycles).

Consider next V_2, another 3-ball with two handles. Glue now the two handle bodies together such that the principal cycles of V_1 are mapped to the cycles given by the unbroken and dotted lines in the picture. In terms of the generators:

$$3C_2 + C_1 + C_2 - C_3 + C_2 - C_4 - C_3 + 2C_4,$$

$$-2C_4 + C_3 - C_2 - C_4 - C_3 + 2C_4 - C_2$$

(additive notation, not commutative in the fundamental group).

First on the level of homologies he reduces by a detailed computation to two generators with two relations:

$$3C_2 + 2C_4 \sim 0,$$

$$-C_4 - 2C_2 \sim 0.$$

This set of linear equations has determinant -1 and so the first Betti number is 1 and there is no torsion. This space has the same homology invariants as the 3-sphere.

Next he shows (again explicit) that the fundamental group is non-zero; generated by C_2 and C_4 with relations

$$-C_2 + C_4 - C_2 + C_4 \equiv 0,\ \ 5C_2 \equiv 0,\ \ 3C_4 \equiv 0.$$

This is the *icosahedral group*. It's commutative image (the first homology group) is trivial. So we have a homology 3-sphere with non-trivial fundamental group!

Poincaré used the cube model to give a presentation of the fundamental group. He started with the 1-skeleton of the cube and added relations according to the two-dimensional faces. Next he computed the Betti numbers (P_1 by abelinization and P_2 by duality):

$$P_1 = P_2 = 2 \text{ in case } (a-1)(d-1) - bc \neq 0,$$

$$P_1 = P_2 = 4 \text{ in case } a = d = 1,\ b = c = 0,$$

$$P_1 = P_2 = 3 \text{ in other cases.}$$

Finally he looked for conditions when two of these fundamental groups are isomorphic. A necessary condition is the conjugation of the two groups. He concluded that there are infinitely many different manifolds with the same Betti numbers.

The Euler–Poincaré characteristic

Euler already showed the formula $V - E + F = 2$ for the number of vertices V, edges E and faces F of a convex polyhedron in $\mathbb{R}^3$. Poincaré generalized this in chapters 16–18 to arbitrary closed manifolds of any dimension p. Given a decomposition in polyhedral cells he looked at the alternating sum of the number of cells of dimension i (denoted by α_i):

$$N = \alpha_p - \alpha_{p-1} \cdots + (-1)^p \alpha_0.$$

Next he showed that N does not change under subdivision. Assuming that polyhedral decompositions always exist and that it is pos-

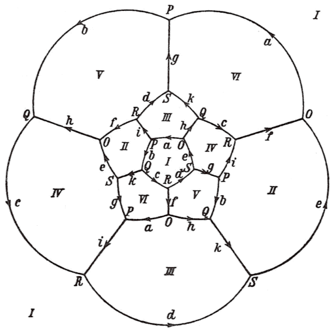

FIG. 112

running around the pentagons we get the following six relations of type (II):

$$\left.\begin{array}{r} ABCDE = 1 \\ BKEF^{-1}J^{-1} = 1 \\ AJDK^{-1}H^{-1} = 1 \\ CJ^{-1}G^{-1}EH = 1 \\ BH^{-1}F^{-1}DG = 1 \\ AG^{-1}K^{-1}CF = 1 \end{array}\right\} \quad \text{or} \quad \left\{\begin{array}{r} BCE = 1 \\ BKEJ^{-1} = 1 \\ J = K \\ CJ^{-1}G^{-1}E = 1 \\ B = G^{-1} \\ G^{-1}K^{-1}C = 1 \end{array}\right.$$

Figure 9 Spherical dodacahedron space

sible to pass to another by a sequence of subdivisions one gets that N does not depend on the polyhedral decomposition. Moreover he showed that N only depends on the Betti numbers of the manifold, so is in fact a homology invariant. In modern language we call the number N the *Euler–Poincaré characteristic* χ. By duality $\chi = 0$ for odd-dimensional manifolds.

The Poincaré sphere

Poincaré asked in [4] the question: “Can two manifolds have the same Betti numbers, but different fundamental groups?” An interesting test case for this is of course the 3-sphere. In Supplement 2 (where he was already aware of torsion) he even “confined himself by stating the following theorem the proof of which will require further developments”:

Each polyhedron, which has all its Betti numbers equal to the Betti numbers of S^3 and has no torsion is homeomorphic to S^3.

Later on in Supplement 5 he disproved this statement via a manifold, which we call now the Poincaré sphere.

He constructed this manifold as follows: Consider two three-dimensional manifolds, in fact handle bodies, with the same surface as boundary and next glue these two together by a diffeomorphism of the boundary.

Note that given any 3-manifold, there exist always a splitting into two such handle bodies: a Heegard decomposition. Poincaré studied these handle bodies (see Figure 7) in detail and showed, that on each handle body there exists a system of so-called principal cycles. For computation of the fundamental group (and homology) of the handle body one can start with the presentation of the boundary surface and add these principal cycles as extra relations.

From a Heegard decomposition with separating surface of genus two Poincaré constructed his homology 3-sphere. See Figure 8. By duality we only have to look to the fundamental group and 1-homologies. He showed (see the explicit computation) that the fundamental group is the *icosahedral group*. Its commutative image (the first homology group) is trivial. So we have a homology 3-sphere with non-trivial fundamental group!

Next he stated his question: “Is it possible to have a 3-manifold with trivial fundamental group which is not diffeomorphic to the 3-sphere.” This became the famous Poincaré conjecture (proved by Perelmann in 2003).

It is nowadays more common to describe the Poincaré sphere by conjugating facets of a regular dodecahedron. This space arises by identifications opposite face with a twist of $\frac{\pi}{5}$. This was checked by Kneser [2] in 1929. See Figure 9 taken from [9]; for details see its page 224.

The Poincaré sphere appears also in the theory of singular hypersurfaces in $\mathbb{C}^3$. It is the intersection of a small 5-sphere around the origin with this singular complex surface $x^2 + y^3 + z^5 = 0$. This type of intersection is called a link of a singularity. Links of $x^p + y^q + z^r = 0$ (p, q, and r pairwise relatively prime positive integers) are known to be homology spheres (named Brieskorn spheres).

Conclusion

In this article we could only describe a restricted part of Analysis Situs and its supplements. There are many aspects left, which I have not touched. There is a discussion about the triangulability of manifolds in Analysis Situs, which is continued in the supplements. It took until 1934 that Cairns proved in full rigour the statement that every differentiable manifold has a polyhedral subdivision. In supplement 3, 4 and 5 there is a description of the topology of algebraic surfaces in $\mathbb{C}^3$. These are four-dimensional spaces. Here one already can see the concepts of monodromy and vanishing cycles, related to complex Morse theory. This seems to be the beginning of singularity theory. Another new subject is the dynamic approach towards an evolution of a manifold from a simpler one (e.g. the n-ball), by attaching a series of handles. This is a beginning of Morse Theory.

Only a small part of Poincaré’s work was devoted to topology. With Analysis Situs he started this new subdicipline in mathematics. Successors of Poincaré were mostly outside France. We mention Brouwer (in Holland), Heegard (in Denmark), Dehn en Hopf (in Germany). The first textbooks appeared in 1930 (*Topology* by Letschetz [3]) and in 1934 (*Lehrbuch der Topologie* by Seiffert and Threllfall [9]). As Lefschetz wrote later: “Perhaps no branch of mathematics did Poincaré lay his stamp more indelibly than on topology.” I refer to the book of Dieudonné [1] for the continuation of this field. There was a lot of work left to make the theory completely rigorous; but there were plenty of idea’s available for future developments.

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