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Poincaré and Brouwer on intuition and logic

In the beginning of the twentieth century the Dutch mathematician Luitzen Egbertus Jan Brouwer published his first papers on intuition and logic. There is no indication that Henri Poincaré was aware of these publications, but it would have been interesting to know what he had have to say about them. In this article Dirk van Dalen, Emeritus Professor Logic and Philosophy of Mathematics, compares the ideas of Poincaré and Brouwer on the foundations of mathematics.

The mathematical foundational landscape at the beginning of the twentieth century was dominated by late nineteenth century novelties, such as symbolic logic, set theory, and formalisation. The generally acknowledged grand master of the Foundations of Mathematics was Henri Poincaré. Not in the sense that he was himself involved in presenting novelties, but rather as a generally acknowledged universal creative mathematician, who could from the height of the Olympus survey, encourage and criticise the developments in the field. This did not mean that he did not actively study a specific more technical subject, but that he left his gifts for others to pursue. There is no doubt that in the first decades of the twentieth century he was the best and most widely read mathematical author. Whole generations of mathematicians were introduced into the intricacies of the foundations of mathematics by Poincaré's Flammarion books.

The purpose of this paper is to look at some specific issues in the œuvre of Poincaré and to compare them with the subsequent ideas of the newcomer L.E.J. Brouwer. As Gerhard Heinzmann and Philippe Nabonnand have already discussed most of the issues at hand in their magisterial paper 'Poincaré: intuitionism, intuition and convention' [14], the present paper can be seen as a footnote to it. I will restrict myself to a few topics that may be of interest.

The comparison of Poincaré and Brouwer will inevitably be somewhat out of focus, as Brouwer's mature foundational papers appeared only after Poincaré's death. There is no indication that Poincaré was familiar with Brouwer's early publications. In particular it is unlikely that he had seen Brouwer's dissertation, written in Dutch, which for a long time was the prime source of Brouwer's intuitionism. There was a contribution of Brouwer in the proceedings of the 1908 Rome conference, a conference that was attended by Poincaré. However, it would be difficult to get a balanced impression from such a condensed report. The surviving correspondence in 1911 between Poincaré and Brouwer deals with automorphic functions and uniformisation [10]. We may safely assume that Poincaré was not aware of Brouwer's 'other life'; hence it remains an open question what Poincaré would had have to say about this new actor on the stage of the foundations of mathematics.

Personal contact between Poincaré and Brouwer remained restricted to a few letters. Brouwer was an admirer of Poincaré, he highly valued Poincaré's work in topology and his contributions to the foundational debate around the century. Poincaré, who knew Brouwer as a topologist, appreciated the newcomer; his reply to a letter of Brouwer on the topic of automorphic functions closed with the sentence: "I am happy to have this opportunity to be in contact with a man of your merit.”

It is clear that Brouwer was thoroughly familiar with most of Poincaré’s papers; the dissertation contains a large number of references to that effect. A conspicuous exception can be found in the correspondence of Brouwer and Hadamard on 24 December 1909 [9], where Hadamard calls Brouwer’s attention to Poincaré’s paper on curves defined by differential equations (published in 1881).

On the whole the ideas of Poincaré and Brouwer show a strong similarity. The reader who consults their foundational publications will however be struck by a striking difference in style. Poincaré published for a large readership in mathematics and physics, and for the cultivated reader in general, as a result his style is literary and pedagogical; he had completely mastered the use of the turn of the century narrative scientific exposition. Brouwer, on the other hand, had no mercy on his readers; he shunned long explanations and indulged in archaic expressions.

About logic and logicians

Both Poincaré and Brouwer were critical of the role of logic in mathematics. There is however a marked difference in their views and reactions. In Poincaré’s writings the work of Russell played a substantial role, Brouwer, on the other hand rejects Russell’s approach of logic on the ground that logical principles hold only for words with a mathematical meaning, “and exactly because Russell’s logic is nothing but a word system, without a presupposed mathematical system to which it applies, there is no reason why no contradictions should appear.” It should not come as a surprise that Russell’s monograph *An Essay on the Foundations of Geometry* is extensively discussed in Brouwer’s dissertation — after all, it deals with mathematics. The last chapter of the dissertation contains a discussion of the various approaches to modern logic, and more or less turns them down on the grounds of his philosophical, constructive views. Poincaré, on the other hand follows Russell’s logical theories with great attention. The difference in outlook between Poincaré and Brouwer is rather striking; Poincaré accepts logic as it is and seeks to safeguard it from the various dangers that had been discovered. In line with the contemporary literature, he attaches great value to the problem of predicativity. The pressing question here is: “Can one define a mathematical object using a class which contains that object?” This indeed is a traditional mathematical practice, used for example in the definition of supremum: the supremum s of a set A of reals is the least number of the class B of all numbers $\geq$ all numbers of A. Obviously, $s \in B$. Such a definition is called *impredicative*. In the logical literature of the beginning of the twentieth century the problem of predicativity plays a major role; the *vicious circle principle* explicitly forbids defining objects in terms of classes containing that object. Poincaré actively took part in analysing predicativity in the context of the Russell paradox, see *Les mathématiques et la logique, La logique de l’infini* [17]. On this issue Brouwer takes an independent position, according to him proofs are mental constructions, and (intuitionistic) logic has its own ‘proof interpretation’ (made precise by Heyting and Kolmogorov). Paradoxes of the Russell type thus ask for a proof construction that cannot be carried out. And thus no ‘intuitionistic truth value’ can be determined. Even in his later papers, where the so-called species are introduced, the predicativity issue is ignored (see also [12, p. 972; 13]).

Reading Poincaré’s many accounts of, and objections to, logic, one gets the impression that he takes a rather ‘physical’ view of the subject. Just as in physical theories, there are external conditions that determine the applicability of logic. In *La Logique de l’ Infini* Poincaré states for example that paradoxes arise because of the application of logic outside its proper domain, i.e. the universe where only sets with finitely many objects occur.

This statement occurs almost literally in Brouwer’s *Intuitionistische Mengenlehre* [3, p. 2]: “In my opinion the Solvability Axiom [also known as ‘Hilbert’s Dogma’] and the principle of the excluded third are both false, and the belief in these dogmas historically is the result of the fact that one at first abstracted classical logic from the mathematics of subsets of a particular finite set, and next ascribed an a priori existence, independent from mathematics, and finally, on the basis of this alleged apriority, applied it to the mathematics of infinite sets.” Brouwer, so to speak, traces the popularity of this dubious principle back to its historical origins. In his Berlin Lectures he offered again his interpretation of the long reign of the ‘superstitious belief in the principle of the excluded third’: “[It] can only be explained by the natural phenomenon, that many objects and mechanisms in the external world with respect to extensive complexes of facts and events can be controlled by considering and treating the system of states of these objects and mechanisms in the space-time world as part of a finite discrete system with finitely many connections between the elements of the underlying system, so that the principle of the excluded third turns out to be tangibly applicable to the relevant complexes of objects and mechanisms.” [6, p. 22]

He was well aware of the fact that the principle of the excluded third could not simply be refuted by logic: “that nonetheless classical mathematics is not right away silenced, is due to the supporting circumstance that although the *principium tertii exclusi* is in fact incorrect, but, as long as one restricts its application to finite groups of properties, it is non-contradictory, so that intuitionism, when fighting the aberrations of classical mathematics, is deprived of the most widely accepted mode of repression of errors of thinking, the *reductio ad absurdum*, and has to rely exclusively on admonition to rational reflection.” [4]

The above-mentioned principle of the excluded third (also called principle of the excluded middle, PEM) is the touchstone for the constructive nature of a theory. It states that any statement A is true or false, in symbols: $A \vee \neg A$ is true. On this Aristotelian principle the important and convenient *proof by contradiction* and *Consistency* ⇔ *Existence* are based. If one takes existence a bit more seriously, then “there is a solution for the equation $A(x) = 0$” means more than “it is impossible that there is no solution”. One wants to produce the number a for which $A(a) = 0$ holds. In Brouwer’s intuitionism *existence* is taken to mean *constructible*, therefore he had to revise logic. He did indeed formulate a constructive interpretation of logic, in particular of the hypothetical judgement [2, p. 125]. The above-mentioned ‘proof interpretation’, where proofs are mental constructions was the basis of a new and stricter logic.

Until the end of his career Brouwer stuck to his fundamental view on the role of logic: “Further there is a system of general rules called *logic*, enabling the subject to deduce from systems of word complexes conveying truths, other word complexes generally conveying truths as well. Causal behaviour of the subject (isolated as well as cooperative) is affected by logic. And again object individuals behave accordingly. This does not mean that the additional word complexes in question convey truths *before* these truths have been experienced, nor that these truths *always can* be experienced. In other words, logic is not a reliable instrument to discover truths and cannot deduce truths which would not be accessible in another way as well." [5] In short: "There are no non-experienced truths and [that] logic is not an absolutely reliable instrument to discover truths."

We see that both Poincaré and Brouwer had their reservations about logic. But their motivation was totally different. In the wake of the paradoxes of Richard and Russell Poincaré saw the problems in logic as technical issues in second- or higher-order logic, shortcomings that could be corrected. Brouwer's scepticism concerned logic *tout court*, already propositional logic was suspect. Consequently Poincaré did not revolutionise logic, he suggested various medicines for the patient, whereas Brouwer completely revised logic on the basis of his thesis "a proof of a statement is a construction". The first steps were taken by Brouwer in his dissertation, where he formulated the underlying idea of the proof interpretation (see [1, 8]). The radical revision of logic paid off in due time, but these first steps required a young radical and not an elderly statesman. Indeed, the failure to deal with the non-effective aspects of logic left the French semi-intuitionistic with a half-hearted program.

In discussions of semi-intuitionism there is always a certain believe or hope that here is the place where constructivism was born. This does not seem justifiable; although certain distinctions were discussed, a wholesale overhaul of mathematics was impossible without a revision of logic. In Poincaré's case a rejection of the constructive tenets is embodied in his slogan: "What does the word existence mean in mathematics? It means freedom of contradiction." (*Les derniers efforts des logisticiens*, [16].) Indeed, it would be hard to imagine a conversion of the prolific Poincaré to a frugal mathematical world of constructivism, but he might very well have recognised Brouwer's mathematics as a viable alternative to the traditional one.

Nonetheless it would have been most illuminating to see his reactions to Brouwer's program; as Couturat could testify, Poincaré was not used to mince words.

Intuition

Poincaré may not have been the first mathematician of the new generation of the end of the century to advocate the restoration of intuition to its legitimate position, but he certainly was the most persistent one. His popular expositions ring with praise of the role of intuition in mathematics, contrasting it in particular with the clerical virtues of logic.

Mathematicians would read his version of intuition mainly as the human capacity to make in mathematical research choices based on an assortment of insights and experiences acquired by the subject. It is indeed this aspect that is highly valued by Poincaré, and probably by almost every mathematician, but there is also the other notion of intuition, called *Anschauung* by Kant. The latter notion is duly discussed by Poincaré and the role of non-Euclidean geometry is discussed in detail, but there it more or less stops.

Moving to Brouwer, we note that in a bold move he posits the so-called *ur-intuition* as the unique basis for mathematics. In one stroke the subject introduces both the discrete (natural) numbers and the continuum, see the rejected parts of Brouwer's dissertation [7, 18]. In later publications the expression 'move in time' is introduced to elucidate the time/continuum intuition. The characteristic of the continuum is expressed in the dissertation as: "Recognising the continuum intuition, the 'flowing', therefore as primitive, as well as the joining in thought of various things as one, which is the basis of any mathematical structure, we can name properties of the continuum as 'matrix of all points' that

Luitzen Egbertus Jan Brouwer (1881–1966) can be thought as a whole.” Since the mathematical continuum is identical with the time continuum, it is interesting that in Brouwer’s notes for the dissertation the creation of the time as matrix of moments, is called a free act of ourselves, and “with that creation at the same time the conditions and all elements for the construction of the whole of mathematics are given; one of these is the three-dimensional Euclidean geometry, and that is a suitable schema to manage in a simple language a group of phenomena, ...”

Here Brouwer and Poincaré share a vision of the continuum as an amorphous, immediately given medium (see [15]). With Poincaré this is an interesting comment on the intuitive character of the continuum, but no further analysis is made. Brouwer did go further by making this intuitive continuum the cornerstone of his mathematics (together with the natural numbers). In the dissertation he turned the intuitive continuum into a measurable continuum which made it amenable to the standard mathematical practice. After his introduction of choice sequences, based on free will in 1918, he could furthermore establish a number of basic properties of the amorphous continuum. For Poincaré these results would have been out of bounds as they were in direct conflict with classical logic.

In spite of the obvious parallels between Poincaré’s and Brouwer’s foundational views, here Poincaré’s and Brouwer’s paths separated. All this must be stated with a serious proviso: Poincaré never saw Brouwer’s new mathematical universe. He may well have strongly objected to the subjective element in intuitionism had he lived longer. But it is equally possible that with his strong intuitions, he would have recognised the viability and legitimacy of choice objects in a revised logical setting.

On the issue of choice elements mathematicians had been very cautious. Non-law-like sequences occur presumably for the first time with Paul DuBois-Reymond [11], they next occur with Borel. Whereas DuBois-Reymond hardly elaborates the underlying ideas, Borel discusses choice sequences in a number of publications. His ultimate conclusion is that the notion is interesting, but does not belong to mathematics proper. In itself this is not surprising, as a convincing treatment of choice objects demands a constructive logic. Hence that road was closed to Borel, and presumably also to Poincaré.

Almost all mathematicians will agree that the castle of mathematics could not be built on a foundation without natural numbers. On this point Poincaré and Brouwer are in full agreement. Their writings show us similar reflections on the topic. The catchwords here are *iteration* and *induction*. If there is any distinction at all, it is that with Poincaré mathematical induction is a prime notion. At various places he proclaims the principle of mathematical induction as ‘a truly synthetic *a priori* judgement’. On the other hand at just as many places he presents iteration as directly given by intuition. He indeed falls back on iteration (or recurrence) to motivate (or prove) mathematical induction. The argument is as natural as it is simple: let $A(n) \to A(n+1)$ be true for all n and let $A(1)$ be true, then, since $A(1) \to A(2)$ is true, also $A(2)$ is true. Now from $A(2)$ is true and $A(2) \to A(3)$ is true, it follows that $A(3)$ is true. By iteration, that is repeating the same operation, one gets that $A(n)$ is true for each n.

A similar effect can be seen in Brouwer’s approach, the difference being that Brouwer accepts *iteration* as immediately given by intuition. In later publications this act is described as the self-unfolding performed by the subject, and immediately provided by intuition. Induction thus becomes a consequence of iteration. In *Science et Hypothèse* Poincaré explicitly expresses the same view: “The power of the mind which knows itself capable of conceiving the unlimited repetition of the same act once this act is possible. The mind has a direct intuition of this power.” We may thus claim that Brouwer and Poincaré were in complete agreement on the role of iteration and induction. Since Poincaré’s *Sur la nature du raisonnement mathématique* goes back to 1894, and was re-issued in *Science et Hypothèse* (1902), it is not unreasonable to guess that Brouwer may have been influenced by Poincaré.

Methodological reflections

The discovery of non-Euclidean geometry, and the interrelations between the various geometries heralded the downfall of the doctrine that our knowledge of space is a priori. And thus the choice of geometry, for example for physical theories, became a matter of convention. Poincaré elaborated the philosophy/methodology of the resulting conventionalism in a large number of publications, for a precise analysis see [14]. To quote just one characteristic statement of Poincaré on the topic: “Next must be examined the frames in which nature seems enclosed and which are called time and space. [...] it is not nature which imposes them upon us, it is we who impose them on nature because we find them convenient.” [15]

Brouwer’s methodology for connecting (parts of) the outer world and suitable theories is based on a different ideology, but results in something rather similar. There are few places where he dwells on this issue, e.g. the dissertation. In the chapter ‘Mathematics and Experience’ Brouwer explains the unexpected success of mathematics in dealing with the natural world. From the intuitionist point of view the outer world consists of the sensations of the subject modulo abstraction under similarity (the technical term is *causal sequence*), i.e. sensations that are similar from a particular point of view are identified, thus yielding *objects*. The resulting system of objects and their relations is then further abstracted to a mathematical system. These mathematical systems are purely abstract conglomerates based on the ur-intuition; they are waiting to be applied. The choice of the mathematical system is up to the subject; he can extend the system to a wider one, which is often useful in simplifying parts of the old one, and which opens up the possibility of ‘prediction’. The subject is free however to revise such extensions, should they conflict with the causal sequences in the outer world (be refuted by experiments). Without going further into Brouwer’s theory of science, which is cloaked in terms of the mental activity of the subject, we see that the relation physics–mathematics (Poincaré) matches the relation outer world–mathematics (Brouwer).

Brouwer’s outer world, which consists for the subject in highly stable or invariant causal sequences (equivalence classes of similar sensations) is after all not that far from Poincaré’s ‘objective reality’: “But what we call objective reality is, in the last analysis, what is common to many thinking beings, and could be common to all; this common part, we shall see, can only be the harmony expressed by mathematical laws.”

The two champions of intuition

Comparing the two grandmasters of topology and the philosophy of mathematics, one is struck by the differences in presentation and in philosophical position. Poincaré’s writings belong to the era of the literary giants of the nineteenth and early twentieth century; he addresses the educated layman as well as the specialist, and cultivates a wonderfully balanced style. The essays of Poincaré on an immense variety of foundational topics almost invariably start from an elementary level, and move up with a wealth of subtle arguments and examples to the issues of the day.

He elaborates most of the issues of the exact sciences and at the same time stresses the ethical and moral aspects that are usually relegated to their place 'between the lines'. The introduction to *La Valeur de La Science* opens with the memorable: "The search for truth should be the goal of our activities; it is the sole end worthy of them." And after that the warns that the search of truth demands utter independence from the individual, whereas we usually derive strength from being united with others: "This is why many of us fear truth; we consider it a cause of weakness. Yet truth should not be feared, for it alone is beautiful." Few expositions of science contain such exhortations — for that reason alone reading Poincaré should be obligatory for students.

His mathematics is also presented in the admirable discourse of the nineteenth century intellectual. The immense popularity of Poincaré's Flammarion books testifies to his considerable gifts as an educator.

There is a striking contrast with Brouwer's policy and style. There is no doubt that Brouwer was equally sincere in his wish to improve, or even save, the world. But where Poincaré cultivated the role of a wise but stern teacher, who knew well that the reader is sooner convinced by an instructive and pleasant discourse, than by a grim sermon presented by an inflexible preacher, Brouwer had no compassion with his audience or readership. Compared to Poincaré he was an old testament prophet who predicted the end of the world, unless ... His admonitions in *Life, Art, and Mysticism* were harsh and uncompromising. The influence of this little read monograph was negligible, but that did not stop Brouwer's efforts to convert the mathematical community with well-chosen and refined attacks on, in his eyes, foolish convictions. The Vienna lecture 'Mathematik, Wissenschaft und Sprache', which was, by the way, Brouwer's first exposition of his philosophy to appear in print, may serve as an example.

Even his mathematical publications were held in awe because of their merciless exactness and parsimony with elucidation; nobody less than Hausdorff complained that "The brevity of Brouwer's papers, which often forces the reader to fill in many details by himself, is most regrettable, in the absence of other impeccable and extensive expositions." The modern reader, however, will be pleasantly surprised with this Bourbaki *avant la lettre* directness of exposition.

Conclusion

Summing up, many of the issues in Poincaré's leisurely expositions, of a strongly methodological nature, reappear in Brouwer's work, be it in a concise and precise way. The decisive step made by Brouwer beyond Poincaré's contributions was his abandoning Aristotelian logic and his switch to a rigorous constructive position, based on the intuition of the subject (including choice sequences). Thus raising the level of the discourse to a higher exactness and precision.

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