Jean Mawhin
Institut de Recherche en Mathématique et Physique
Université Catholique de Louvain
Chemin du Cyclotron 2
B-1348 Louvain-la-Neuve, Belgium
jean.mawhin@uclouvain.be

Henri Poincaré and partial differential equations

Henri Poincaré introduced a new approach to solve the Dirichlet problem and he gave the first general solution of the initial value problem for the 'telegraph equation'. In this article Jean Mawhin describes these and other contributions of Poincaré to partial differential equations.

Henri Poincaré has been professor of mathematical physics and probabilities from 1886 until 1896, when he exchanged this chair for that of theoretical astronomy and celestial mechanics left vacant by Félix Tisserand's unexpected death. The story of the surprising nomination of a specialist in number theory and analysis at this chair of mathematical physics is told in [2]. All Poincaré's papers about partial differential equations have been published after 1886. Many of his original contributions or presentations are contained in the well-known series of textbooks based upon his lectures at the Sorbonne.

Between 1890 and 1896, Poincaré devoted three long and important *memoirs* [26, 29, 33] to the partial differential equations of mathematical physics. Among the achievements in those papers are the first general existence result for the Dirichlet problem associated to the Laplacian on a general bounded domain, an implicit minimax characterization of its eigenvalues, an important inequality and the first existence proof of all eigenvalues for the same problem.

Other significant contributions of Poincaré to partial differential equations deal with the first general solution of the telegraph equation on an infinite line in 1893, and the use of a continuation method for the solution of some non-linear elliptic problems. We briefly analyse them in this paper but, to keep it within a reasonable size, we have excluded other Poincaré contributions to partial differential equations, like his papers dealing with the propagation and diffraction of Hertzian waves.

Dirichlet problem and sweeping out method

Given a domain $D \subset \mathbb{R}^3$ bounded by a surface S, and a continuous function Φ over S, a *Dirichlet problem* consists in finding a function V such that

$$\begin{aligned} &\Delta V := \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad \text{in } D, \\ &V = \Phi \quad \text{on } S. \end{aligned} \tag{1}$$

If $\nabla U := (\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z})$, any minimizer V of the integral $\mathcal{I}(U) := \int_D \|\nabla U\|^2$ over all functions $U \in C^2(D) \cap C(\overline{D})$ equal to Φ on S is a solution of (1). Bernhard Riemann called the argument *Dirichlet's principle*, used before him by Johann Peter Lejeune Dirichlet and William Thomson, claiming the existence of a minimum from the positivity of $\mathcal{I}(U)$. After Karl Weierstrass' criticism, several mathematicians of the end of the nineteenth century, like Carl Neumann and Gustave Robin, searched for other existence proofs for the Dirichlet problem, which, according to Poincaré, "are methods of proof of a solution of the problem and computational methods to solve it effectively. As methods of proof, they are rather complicated, but they complete each other in order to apply to all cases and to satisfy the most severe judges with respect to rigour. As methods of computation, they have no value; even the sim-

Johann Dirichlet (1805–1859)

plest of them [...] leads to inextricable computations from the second approximation.”

In [26], Poincaré introduced a new approach to solve the Dirichlet problem in bounded domains $D \subset \mathbb{R}^3$ with sufficiently smooth boundary S. He gave a modest and lucid appreciation of his method: “The most interesting thing would have been to replace the present methods of computations by less defective ones. I was unable to do it, and restricted myself to find a method of proof simpler than the ones proposed until now, and directly applicable to all cases.”

The main ideas of his *sweeping out method* were announced on 3 January 1887 in a note of the *Comptes Rendus* [23], and developed in [26] for the exterior Dirichlet problem. Poincaré applied it directly to the Dirichlet problem (1) in his book [34], and we describe it here in this more familiar frame. The *sweeping out of a sphere* S consists in replacing matter (or charge) inside of S by an equivalent distribution of matter (or charge) on S. In this way, a material point M of mass one can be replaced by a mass one distributed on the boundary with a superficial density inversely proportional to the cube of the distance to M, and the result is easily extended to an arbitrary mass distribution inside S. This process does not introduce any negative mass, does not modify the potential outside S and diminishes it inside S.

Approximating Φ through a series of polynomials on a closed ball B containing D in its interior, and using the maximum principle and Harnack’s theorem, Poincaré showed that *it suffices to solve the Dirichlet problem for data which are the restriction to S of a polynomial P*. Poincaré first expressed D as the union of a sequence of balls (B_n) contained in D, and assumed that $\Delta P < 0$ on B. Letting $\Delta P = -4\pi\sigma$ and $W_0 = \int_B \frac{\sigma}{r}\, dx$, in such a way that $\Delta W_0 = -4\pi\sigma = \Delta P$ in B, and looking at σ as a density of attracting matter, Poincaré swept out the balls B_n in the order

$$B_1, B_2; B_1, B_2, B_3; B_1, B_2, B_3, B_4; \ldots,$$

so that every ball is swept out an infinity of times. If W_n denotes the potential of the attracting matter after the nth operation, it is clear that $W_n > 0$ and that $W_n \le W_{n-1}$. Consequently, the sequence (W_n) converges pointwise, say to W, with $0 < W < W_0$ in D and $W = W_0$ outside of D. If the ball B_k has been swept out during the operations numbered by $\alpha_1, \alpha_2, \alpha_3, \ldots$, the functions W_{α_j} are harmonic in B_k, and the sequence (W_{α_j}) converges to W. From Harnack’s theorem, W is harmonic in B_k, and hence in $D = \cup_{k\ge1} B_k$.

Poincaré then showed that *W is continuous at any point Q of S in which S has a tangent plane and two principal curvature radii.* The function $V = W - W_0 + P$ is harmonic in D since $\Delta W = 0$ and $\Delta W_0 = \Delta P$, and is equal to P on S, since $W = W_0$ on S. The Dirichlet problem is therefore solved when $\Delta P < 0$ and S has the mentioned regularity at each point. The restriction $\Delta P < 0$ can be easily removed because any polynomial P can be written as the difference $P_2 - P_1$ of two polynomials such that $\Delta P_1 < 0$ and $\Delta P_2 < 0$. If V_j is the harmonic function in D equal to P_j on S $(j = 1, 2)$, then $V = V_1 - V_2$ is the searched function.

The regularity conditions upon the boundary have been generalized by Henri Lebesgue and by Norbert Wiener. A short account of Poincaré’s work is given in [6]. More details can be found in [16] and [40]. It is superfluous to insist over the consequences and developments of the sweeping out method in potential theory and in the study of elliptic equations, in the hands of mathematicians like Stanislaus Zaremba, Henri Lebesgue, Oskar Perron, Norbert Wiener, Frédéric Riesz, Otto Frostman, Charles de La Vallée Poussin, Henri Cartan, Marcel Brelot and others [5, 19].

Variational characterization of the eigenvalues of Fourier’s problem

Separation of space and time dependence in *Fourier’s problem for heat equation* leads to the determination of positive numbers k and functions U such that

$$\Delta U + kU = 0 \quad \text{in } D \subset \mathbb{R}^3,$$
$$\frac{\partial U}{\partial n} + hU = 0 \quad \text{on } S, \quad \int_D U^2 = 1,$$

with $D \subset \mathbb{R}^3$ a bounded domain with boundary S, $\frac{\partial U}{\partial n}$ the exterior normal derivative, and h a positive constant. For special domains D, the problem can be explicitly solved, but for an arbitrary bounded D, Poincaré observed that “...the first point is to establish the existence of those functions U. This has not been done yet, as far as I know, in the general case, and I will try to do it.”

The first approach to this problem, also contained in [26], was announced in two notes of the *Comptes Rendus* from 20 June 1887 [24] and 17 December 1888 [25]. Poincaré was aware of their lack of rigour, but not of their lack of novelty. A previous work of Heinrich Weber in 1869 [43] had proposed an entirely similar approach in dimension two. The underlying idea of both works is an extension of Dirichlet’s principle. If $B(U_1)$ denotes the minimum of the positive quadratic expression

$$B(F) := h \int_S F^2 d\sigma + \int_D \|\nabla F\|^2$$

on $\Sigma_1 := \{F \in C^1 : \int_D F^2 = 1\}$, there is a Lagrange multiplier k_1 such that

$$\Delta U_1 + k_1 U_1 = 0 \quad \text{in } D \subset \mathbb{R}^3,$$
$$\frac{\partial U_1}{\partial n} + hU_1 = 0 \quad \text{on } S.$$

The existence of a *fundamental function* U_1, in Poincaré's terminology, is 'proved' in this way, and Poincaré observed that

$$k_1 \leq \frac{h\int_S F^2 d\sigma + \int_D \|\nabla F\|^2}{\int_D F^2},$$

for all non-zero functions F.

Similarly, if $B(U_2)$ denotes the minimum of $B(F)$ on $\Sigma_2 \subset \Sigma_1$ defined by $\Sigma_2 := \{F \in C^1 : \int_D F^2 = 1,\ \int_D FU_1 = 0\}$, there are Lagrange multipliers k_2 and λ such that

$$\Delta U_2 + k_1 U_2 + \lambda U_2 = 0 \quad \text{in } D \subset \mathbb{R}^3,$$
$$\frac{\partial U_2}{\partial n} + hU_2 = 0 \quad \text{on } S.$$

The conditions $\int_D U_1^2 = 1$, $\int_D U_1 U_2 = 0$ and Green's formulas imply that $\lambda = 0$, and hence U_2 is a second fundamental function. If we now minimize F on $\Sigma_3 = \{F \in C^1 : \int_D F^2 = 1,\ \int_D FU_1 = 0,\ \int_D FU_2 = 0\}$, we obtain the existence of some number k_3 and of a third fundamental function U_3, and, going on in this way, the existence of infinitely many fundamental functions. Poincaré admitted the non-rigorous character of his reasoning: "This proof is completely analogous to the one used by Riemann to establish Dirichlet's principle, that analysts have since replaced by more rigorous reasoning."

Poincaré was one of them, with his sweeping out method. Poincaré went further than Weber in obtaining an *upper bound* for k_n. Given $F_1, \ldots, F_n$, if $F = \sum_{j=1}^n \alpha_j F_j$, then $B(F)$ and $A(F) := \int_D F^2$ are positive definite quadratic forms in $\alpha = (\alpha_1, \ldots, \alpha_n)$ and a simple algebraic reasoning shows that the roots $\lambda_1 \leq \cdots \leq \lambda_n$ of the characteristic equation $|B(F) - \lambda A(F)| = 0$ verify the inequalities $k_j \leq \lambda_j$ $(1 \leq j \leq n)$. As George Polya noticed later [39], if one takes $F_j = U_j$ $(1 \leq j \leq n)$, then $k_j = \lambda_j$ $(1 \leq j \leq n)$, and, if $S_n = \{\sum_{j=1}^n \alpha_j F_j : \alpha \in \mathbb{R}^n\}$, then $\lambda_n = \max_{F \in S_n \setminus \{0\}} B(F)/A(F)$. Consequently, Poincaré's reasoning *implicitly* contained the *minimax characterization of* k_n,

$$k_n = \min_{S_n} \max_{F \in S_n} \frac{B(F)}{A(F)},$$

explicitly given for the first time in 1905, for a finite-dimensional spectral problem, by Ernst Fisher [8], without any reference and through an algebraic approach independent of the calculus of variations. It is not necessary to insist on the importance and the development of minimax methods for the study of eigenvalues, by Herman Weyl, Richard Courant, Alexander Weinstein, George Polya, Menahem Schiffer and many others [10, 44].

Using once more Green's formula, Poincaré showed the increasing character of the k_j with respect to h. As, by construction, they also increase with j, it suffices to treat the case where $h = 0$, i.e. *Neumann's problem,* to show that $k_j \to +\infty$ as $j \to +\infty$. To this aim, Poincaré searched a *lower bound* for k_n. Decomposing D in p parts D_i $(1 \leq i \leq p)$, calling $U_{i,j}$ and $k_{i,j}$ the fundamental functions and the characteristic numbers associated to Neumann conditions on D_i, and letting $V = \sum_{l=1}^n \alpha_l U_l$, where the U_l are the first n characteristic functions on D and the α_l undetermined coefficients, it is easy to see that

$$\frac{\int_D \|\nabla V\|^2}{\int_D V^2} = \frac{\sum_{l=1}^n k_l \alpha_l^2}{\sum_{l=1}^n \alpha_l^2} \leq k_n. \tag{2}$$

Choosing the coefficients in such a way that $\int_{D_i} VU_{i,1} = 0,\ \ j = 1, \ldots, \lambda_i,\ \ i = 1, \ldots, n-1$, Poincaré obtained easily that

$$\int_D \|\nabla V\|^2 = \sum_{i=1}^{n-1} \int_{D_i} \|\nabla V\|^2 \geq \min_{1 \leq i \leq n-1} \{k_{i,2}\} \int_D V^2,$$

and hence, using (2), that $k_n \geq \min_{1 \leq i \leq n-1}\{k_{i,2}\}$. In particular, a polyhedron bounded by faces parallel to one of the planes of coordinates can be decomposed into $n-1$ rectangular parallelotopes whose largest side tends to zero when $n \to \infty$. If those largest sides are all smaller or equal to a_n, one obtains, by separation of variables, $k_{i,2} \geq \pi^2/a_n^2$ $(i = 1, \cdots, n-1)$, and hence $k_n \geq \pi^2/a_n^2$, so that $k_n \to +\infty$ when $n \to \infty$. If one can approach D by such polyedra the result remains valid for D. But Poincaré did not content himself with this reasoning and proposed to find a lower bound to k_2 when D is a bounded convex domain. This is the origin of the famous *Poincaré inequality*, to which the next section is devoted.

Poincaré inequality

Let $D \subset \mathbb{R}^3$ be a bounded and convex open set. Poincaré first observed that, with $\mu(D)$ the volume of D,

$$\int_{D \times D} [V(x) - V(x')]^2 \, dx\, dx' = 2\mu(D) \int_D V^2 - 2\left(\int_D V\right)^2.$$

Therefore

$$\begin{aligned} k_2 &= 2\mu(D) \min_{V \neq 0: \int_D V = 0} \frac{\int_D \|\nabla V\|^2}{\int_{D\times D}[V(x) - V(x')]^2\, dx\, dx'} \\ &= 2\mu(D) \min_{V \neq 0} \frac{\int_D \|\nabla V\|^2}{\int_{D\times D}[V(x) - V(x')]^2\, dx\, dx'}, \end{aligned} \tag{3}$$

as the minimized expression does not change by adding a constant to V. Using a complicated argument based upon a spherical type change of variable, on which we return later, and upon a reduction to a one-dimensional problem of the calculus of variations, Poincaré obtained, *for continuously differentiable functions* V *such that* $\int_D V = 0$, the inequality

$$k_2 \geq \frac{6K_0 \mu(D)}{\pi d^5}, \tag{4}$$

where K_0 is some numerical constant and d the diameter of D.

This first formulation of the *Poincaré inequality* allowed him to extend his result $\lim_{n\to\infty} k_n = +\infty$ to the case of a 'general' domain. If one decomposes D in $n-1$ convex parts D_j $(1 \leq j \leq n-1)$, and if $\mu(D_j)/d_j^5 \geq \alpha$ for any D_j and some $\alpha > 0$, one easily gets $k_n \geq 6K_0\alpha/\pi$. The quantity α (of the order of d_j^{-2}) can be taken arbitrarily large by taking n sufficiently large and choosing suitably the parts D_j. Hence, for such a domain D, $k_n \to \infty$ when $n \to \infty$.

Poincaré returned to inequality (4) in the third section 'Preliminary lemma' of his second memoir [29] on the equations of mathematical physics. He substantially simplified his argument and obtained a more explicit expression than in (4). For simplicity, we restrict to the case of a bounded convex open set D of $\mathbb{R}^2$, for which Poincaré only stated the result. To estimate the right-hand member of (3), Poincaré introduced, like in the preceding memoir, the change of variables

$$\begin{aligned} x &= \xi + \rho\cos\varphi, \quad y = \rho\sin\varphi, \\ x' &= \xi + \rho'\cos\varphi, \quad y' = \rho\sin\varphi. \end{aligned} \tag{5}$$

Geometrically, ξ is the first component of the intersection of the line joining (x,y) to (x',y') with the first axis, φ the angle of the segment joining $(\xi,0)$ to (x,y) with this axis, ρ the distance of (x,y) to $(\xi,0)$ and ρ' the distance of (x',y') to $(\xi,0)$. The aim of this change of variables is essentially to reduce the inequality to the one-dimensional case. Letting $T(\xi,\varphi,\sigma) = (\xi+\sigma\cos\varphi, \sigma\sin\varphi)$, and using the Cauchy inequality, Poincaré obtained the estimate

$$\left[\frac{\partial}{\partial\rho}(V\circ T)\right]^2 \leq \left(\frac{\partial V}{\partial x}\circ T\right)^2 + \left(\frac{\partial V}{\partial y}\circ T\right)^2 = \|\nabla V\circ T\|^2. \tag{6}$$

If $\mu(D)$ denotes the area of D, using (6), one has

$$\mu(D)\int_D \|\nabla V\|^2 = \int_{D\times D} \|\nabla V(x)\|^2\,dx\,dx'. \tag{7}$$

The idea consists in using the transformation (5) to compute the right-hand member integral as well as $\int_{D\times D}[V(x)-V(x')]^2\,dx\,dx'$, to notice that, for $\rho' > \rho$, the Schwarz inequality implies, with $T' = T(\xi,\varphi,\sigma')$,

$$(V\circ T - V\circ T')^2 \leq (\rho'-\rho)\int_\rho^{\rho'}\left[\frac{\partial}{\partial\sigma}(V\circ T)\right]^2 d\sigma,$$

integrate this inequality over the other variables and use (6) to obtain the *Poincaré inequality in the plane*

$$\int_D \|\nabla V\|^2 \geq \frac{24}{7d^2}\int_D V^2, \tag{8}$$

for any continuously differentiable function V such that $\int_D V = 0$.

When D is a bounded convex open subset of, a similar approach gives the *Poincaré inequality in space*

$$\int_D \|\nabla V\|^2 \geq \frac{16}{9d^2}\int_D V^2 \tag{9}$$

for all continuously differentiable functions V such that $\int_D V = 0$. Modern treatments of the Poincaré inequality only prove, using a compactness argument, the *existence* of a constant $c > 0$ such that

$$\int_D \|\nabla V\|^2 \geq c\int_D V^2 \tag{10}$$

when $\int_D V = 0$, without information about the dependence of c with respect to D. A recent work [3] shows that the best constant c for a convex $D \subset \mathbb{R}^n$ is equal to $\frac{\pi^2}{d^2}$ for all n. On the other hand, the inequality (10) for the functions *which vanish on S*, often referred as the Poincaré inequality in the literature, does not occur in the work of the French mathematician, but can be traced to Herman A. Schwarz.

Poincaré uses inequalities (8)–(9) to show that, given p functions $\varphi_1,\ldots,\varphi_p$ and $V = \sum_{i=1}^p \alpha_i\varphi_i$, one can choose the α_i in such a way that $\int_D \|\nabla V\|^2/\int_D V^2$ is larger than any number given in advance. First, if D can be decomposed into $p-1$ convex open sets D_i of diameters smaller than d, choosing the α_i in order that $\int_{D_i} V = 0 \quad (1 \leq i \leq p-1)$, inequality (9) applied to D_i implies that

$$\int_D \|\nabla V\|^2 = \sum_{i=1}^{p-1}\int_{D_i}\|\nabla V\|^2 \geq \frac{16}{9d^2}\sum_{i=1}^{p-1}\int_{D_i} V^2 = \frac{16}{9d^2}\int_D V^2.$$

If now D *is convex* and contained in a cube of side Λ, one divides this cube into q^3 equal cubes of side Λ/q. The intersection of such a small cube with D is convex and has diameter smaller than $\Lambda\sqrt{3}/q$. If one has $p-1$ intersections, the reasoning above shows that one can choose the α_i in such a way that

$$\int_D \|\nabla V\|^2 \geq \frac{16q^2}{27\Lambda^2}\int_D V^2.$$

It will be possible to do it *a fortiori* if p is larger than the number of intersections plus one, and it suffices to take $p \geq q^3+1$. Consequently, *for D convex, one can choose the α_i in such a way that*

$$\int_D \|\nabla V\|^2 \geq L_p\int_D V^2 \tag{11}$$

with $L_p = \frac{16q^2}{27\Lambda^2}$ and q^3 the largest perfect cube contained in $p-1$. If D can be decomposed into m convex sets, one still obtains the inequality (11) for a suitable choice of the α_i, with q^3 the largest perfect cube contained in $\frac{p-1}{m}$. Similar arguments work in dimension 2, and, when $p\to\infty$, $L_p \simeq p^{2/3}$ if $n=3$ and $L_p \simeq p$ if $n=2$.

More details on Poincaré's contributions to his inequality can be found in [1] and [16].

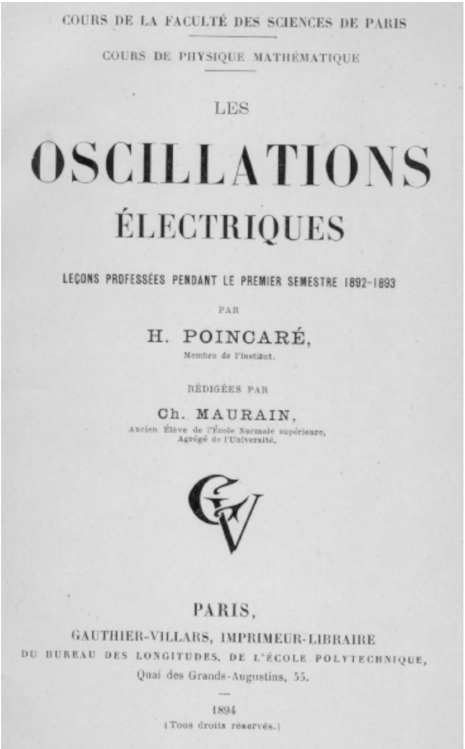
Eigenvalues and eigenfunctions of the Dirichlet problem

One of the main contributions of the memoir [29] of 1894 is to provide the first *rigorous* proof of the existence of eigenvalues and eigenfunctions for the Dirichlet problem in an arbitrary bounded domain. The existence of the first eigenvalue had been shown in 1885 by Herman A. Schwarz [41] and that of the second one in 1893 by Émile Picard [22]. Poincaré's result was announced in a note in the *Comptes Rendus* of 26 February 1894 [28].

Given a bounded domain $D \subset \mathbb{R}^3$ with boundary S, a continuous function f over D, and $\xi \in \mathbb{R}$, Poincaré's idea was to start by the study of the *non-homogeneous* Dirichlet problem

$$\Delta v + \xi v + f = 0 \quad \text{in } D, \quad v = 0 \quad \text{on } S, \tag{12}$$

and to detect the eigenvalues as the values of ξ for which the solution of (12) is infinite, i.e. for which *resonance* occurs. Following a method of Schwarz, he searched the solution in the form of a series $v = \sum_{k=0}^\infty \xi^k v_k$, leading formally to the sequence of Dirichlet problems



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PAR

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Membre de l'Institut.

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Ch. MAURAIN,

Ancien Élève de l'École Normale supérieure, Agrégé de l'Université.

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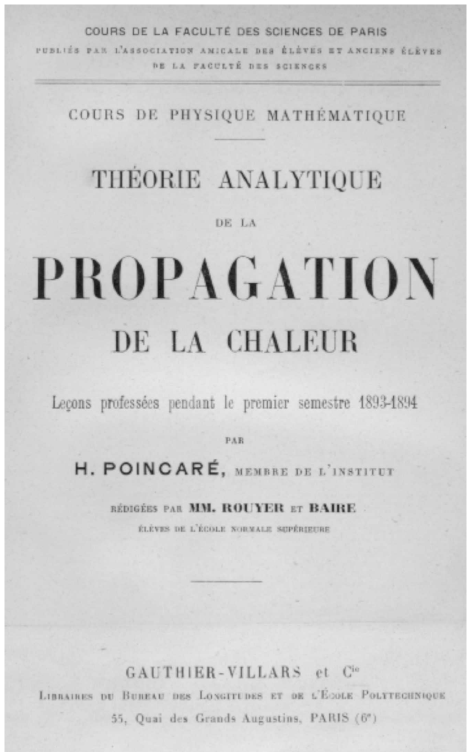

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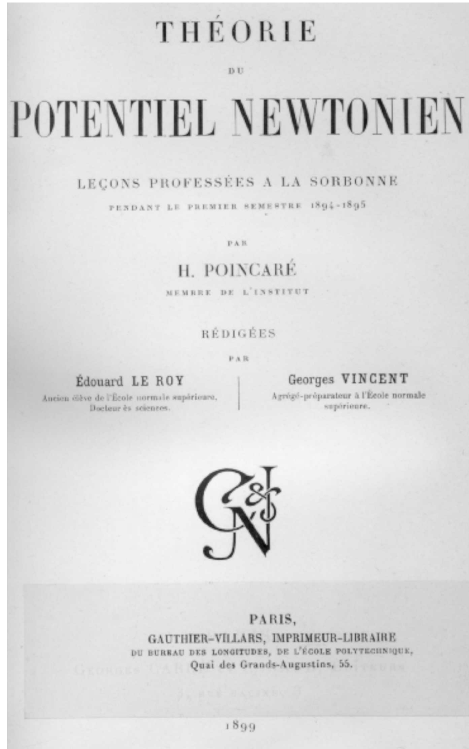

THÉORIE

DU

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LEÇONS PROFESSÉES A LA SORBONNE

PENDANT LE PREMIER SEMESTRE 1894-1895

PAR

H. POINCARÉ

MEMBRE DE L'INSTITUT

RÉDIGÉES

PAR

Édouard LE ROY

Ancien élève de l'École normale supérieure, Docteur ès sciences.

Georges VINCENT

Agrégé-préparateur à l'École normale supérieure.

PARIS,

GAUTHIER-VILLARS, IMPRIMEUR-LIBRAIRE

DU BUREAU DES LONGITUDES, DE L'ÉCOLE POLYTECHNIQUE,

Quai des Grands-Augustins, 55.

1899

Front covers of some of Poincaré's publications [30–31, 34]

for the Laplacian

$$\begin{aligned} \Delta v_0 + f = 0 \quad \text{in } D, \quad v_0 = 0 \quad \text{on } S, \\ \Delta v_k + v_{k-1} = 0 \quad \text{in } D, \quad v_k = 0 \quad \text{on } S \ (k = 1, 2, \ldots), \end{aligned}$$

each of which having a unique solution by the sweeping out method. Using the 'Schwarz' integrals'

$$W_{m,n} := \int_D v_m v_n, \quad V_{m,n} := \int_D \nabla v_m \cdot \nabla v_n (m, n = 0, 1, 2, \ldots),$$

which are such that $W_{m,n} = W_{m+n,0} := W_{m+n}$, Poincaré showed that the sequence (W_{n+1}/W_n) converges to some $1/R$ such that $R \geq Q\sqrt{\mu(D)}$. This implies the uniform convergence (with respect to x) of the series $\sum_{n=0}^{\infty} \xi^n v_n(x)$ for $|\xi| < R$, and that *problem (12) has a solution* $v = [f, \xi]$ *for any* $|\xi| < R$, *and in particular for any* $|\xi| < Q\sqrt{\mu(D)}$.

If φ_j $(1 \leq j \leq p)$ are p given functions, $w_j = [\varphi_j, \xi]$ $(1 \leq j \leq p)$, $\varphi = \sum_{j=1}^{p} \alpha_j \varphi_j$, $v = \sum_{j=1}^{p} \alpha_j w_j$, then, by linearity, $v = [\varphi, \xi] = \sum_{n=0}^{\infty} \xi^n v_n$. Using an argument of nested closed sets, Poincaré showed that the α_j can be chosen in such a way that the corresponding R is larger or equal to L_p, where L_p is given by (11). To apply this observation to problem (12), let

$$v = [f, \xi] = \sum_{n=0}^{\infty} \xi^n v_n, \quad u_j = [v_{j-2}, \xi] \quad (2 \leq j \leq p).$$

If $u_j = \sum_{n=0}^{\infty} \xi^n u_{j,n}$, one has, with the zero Dirichlet condition on the boundary,

$$\Delta u_{j,0} + v_{j-2} = 0, \Delta u_{j,1} + u_{j,0} = 0, \ldots, \Delta u_{j,n} + u_{j,n-1} = 0, \ldots$$

and hence, by uniqueness of Dirichlet problem,

$$u_{j,0} = v_{j-1}, u_{j,1} = v_j, \ldots, u_{j,n} = v_{j+n-1}, \ldots$$

i.e. $u_j = \sum_{n=0}^{\infty} \xi^n v_{j+n-1}$ $(2 \leq j \leq p)$. Let now $w = \alpha_1 v + \sum_{j=2}^{p} \alpha_j u_j$. By linearity,

$$\begin{aligned} w = \sum_{n=0}^{\infty} \xi^n w_n &= \left[\alpha_1 v + \sum_{j=2}^{p} \alpha_j u_j, \xi \right] \\ &= \sum_{n=0}^{\infty} \xi^n \left(\alpha_1 v_n + \sum_{j=2}^{p} \alpha_j v_{j+n-1} \right). \end{aligned} \tag{13}$$

Using the result above, one can choose the α_j in such a way that the series (13) has a radius of convergence at least equal to L_p. Identifying the coefficients of the two power series for w and using Cramer's rule, one gets the expression $v = P(x, y, z, \xi)/D(\xi)$, with $D(\xi) = \sum_{j=0}^{p-1} (-1)^j \alpha_{p-j} \xi^j$ and $P(x, y, z, \xi)$

$$= \begin{vmatrix} w(x,y,z,\xi) & \alpha_2 & \alpha_3 & \cdots & \alpha_{p-1} & \alpha_p \\ v_0(x,y,z) & -\xi & 0 & \cdots & 0 & 0 \\ v_1(x,y,z) & 1 & -\xi & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ v_{p-2}(x,y,z) & 0 & 0 & \cdots & 1 & -\xi \end{vmatrix}.$$

As $v_0, v_1, \ldots, v_{p-2}$ do not depend upon ξ and w is holomorphic in ξ for $|\xi| < L_p$, the function $v = P/D$ is meromorphic in ξ for $|\xi| < L_p$. As we can take p as large as we want, v *is meromorphic in* ξ *in the whole complex plane*. It is not difficult to see that $P(\cdot, \xi)$ vanishes on S, has partial derivatives of the first and second order with respect to x, y, z, and satisfies the partial differential equation

$$\Delta P = \Delta(vD) = D\Delta v = -\xi Dv + Df = -\xi P + Df. \qquad (14)$$

The poles of modulus smaller or equal to L_p of the meromorphic function v are the zeros of the polynomial D. If $\xi = k$ is a simple zero of D, and $P_k = P(\cdot, k)$, then (14) implies $\Delta P_k + kP_k = 0$, which shows that k is a *characteristic number* associated to the *harmonic function* P_k, i.e., in present terminology, P_k is an *eigenfunction* of $-\Delta$ with Dirichlet conditions upon S, associated to the *eigenvalue* k. Poincaré justified as follows his obsolete terminology: "The various simple sounds that a membrane can produce are characterized by equations of the form $\Delta u + ku = 0$, the function being requested to vanish at the boundary. It is well known that those simple sounds have been called harmonics."

Poincaré showed then that v *only admits simple poles,* and hence, up to a multiplicative constant, P_k *is the residue of* v *at* k. Through relatively elementary considerations, he also showed that v cannot be holomorphic in the whole plane, which insures the existence of harmonic functions, that one cannot have more than $p-1$ linearly independent harmonic functions linearly with characteristic number smaller than L_p, and hence that *to any characteristic number can correspond only a finite number of linearly independent harmonic functions,* and finally that *all the characteristic numbers are real and positive.* Poincaré *normed* the harmonic functions by the condition $\int_D P^2 = 1$.

Poincaré's clever proof has been rapidly forgotten, because of the development, by Ivar Fredholm et David Hilbert, of the *spectral theory of linear integral equations,* that it had strongly inspired. Informations about the developments of the spectral theory in the twentieth century can be found in [7], and more details on Poincaré's contributions are given in [16] and [40].

The next chapter of the memoir [29] studies the *maximum principle* for problem

$$\Delta v = -f \ \text{ in } D, \quad \frac{\partial v}{\partial n} + hv = \varphi \ \text{ on } S, \qquad (15)$$

when $f > 0$, $h > 0$ and $\varphi > 0$. Its most original contribution is the introduction of the idea of *weak solution* of problem (15). Poincaré observed that if v is a solution of (15) and u is an arbitrary function twice continuously differentiable, then the relation

$$\int_D uf\,dx + \int_D v\Delta u\,dx + \int_S v\varphi\,d\sigma = \int_S v\left(hu + \frac{\partial u}{\partial n}\right)d\sigma, \qquad (16)$$

that Poincaré called the *modified condition,* holds. Conversely, if condition (16) holds for any u, the equations (15) are also satisfied, provided v and $\frac{\partial v}{\partial n}$ are continuous. Poincaré then showed that if v is just continuous and satisfies (16) for all u, the maximum principles developed in the classical frame remain valid. After having recalled the idea of Neumann's method for the Dirichlet problem on which we will come back later, Poincaré applied it to the mixed problem

$$\Delta v + f = 0 \ \text{ in } D, \quad \frac{\partial v}{\partial n} + hv = 0 \ \text{ on } S,$$

when D is a bounded convex open set. He did not succeed in proving that the solution v obtained by this method of successive approximations has a normal derivative on S and, *a fortiori*, that it satisfies the boundary condition. But he proved that it satisfies the condition (16) (with $\varphi = 0$), anticipating again the concept of weak solution when writing: "We are not sure that the expression $\frac{\partial v}{\partial n}$ has a meaning and that the boundary condition $\frac{\partial v}{\partial n} + hu = 0$ is fulfilled. But we can assert that we have the *modified condition*. It is obviously physically equivalent."

Finally, the remaining part of the memoir was a tentative extension of his proof for the existence of the eigenvalues to the problem

$$\Delta v + \xi v + f = 0 \ \text{ in } \Omega, \quad \frac{\partial v}{\partial n} + hv = 0 \ \text{ on } S,$$

which met the same difficulty concerning the regularity of the coefficients v_n on the boundary. Poincaré left the complete solution to his followers: "One can see that I have not been able to obtain in the general case results as satisfactory as in the case $h = \infty$; one can see how many gaps still remain. I will not try to fill them more [...]. I think however that those results, even incomplete, were not completely without interest, and I have decided to publish them. I would be happy if this publication could suggest new researches on this topics."

No doubt that Poincaré's desire has been fully realized. The memoir ends with some partial results on the expansion of a function in series of harmonic functions.

Neumann's method for the Dirichlet problem on an arbitrary domain

In 1878, Carl Neumann [20] had justified mathematically a formal method, introduced by Beer around 1860, to obtain the existence of a solution to the Dirichlet problem

$$\Delta V = 0 \ \text{ in } D, \quad V = \Phi \ \text{ on } S, \qquad (17)$$

when Φ is continuous and D is bounded, convex, with sufficiently regular boundary S. The method consisted in searching V in the form of a *double layer potential* of unknown density ρ, i.e. as a surface integral

$$V(x) = \int_S \rho(s)\frac{\partial}{\partial n}\frac{1}{\|x-s\|}\,d\sigma.$$

It is well known that V is harmonic outside S, and that, for $y \in S$, the limits

$$V^i(y) = \lim_{x\to y; x\in D} V(x), \quad V^e(y) = \lim_{x\to y; x\notin \overline{D}} V(x)$$

exist and satisfy $V^i(y) - V^e(y) = -4\pi\rho(y)$. Letting, for $y \in S$, $V(y) = \frac{1}{2}[V^i(y) + V^e(y)]$, Neumann tried to determine V in such a way that, on S, the relation

$$V^i - V^e = \lambda(V^i + V^e) + 2\Phi \qquad (18)$$

holds for some real parameter λ. If the problem is solved for $\lambda = -1$, then $V^i = \Phi$ on S, and V is a solution of (17). Searching V as a series $V = \sum_{n=0}^{\infty}\lambda^n V_n$, which gives $V^i = \sum_{n=0}^{\infty}\lambda^n V_n^i$, $V^e = \sum_{n=0}^{\infty}\lambda^n V_n^e$, and, on S, $V_n = \frac{1}{2}(V_n^i + V_n^e)$, one finds, identifying in (18) the coefficients of the similar powers of λ, the recurrence relations, for $x \in S$,

$$\begin{aligned} V_0(x) &= -\int_S \frac{\Phi(s)}{2\pi}\frac{\partial}{\partial n}\frac{1}{\|x-s\|}\,d\sigma, \\ V_n(x) &= -\int_S \frac{V_{n-1}(s)}{2\pi}\frac{\partial}{\partial n}\frac{1}{\|x-s\|}\,d\sigma \ (n = 1, 2, \ldots). \end{aligned} \qquad (19)$$

When D is convex, Neumann proved the existence of a constant C such that the series $\sum_{n=0}^{\infty} \lambda^n (V_n - C)$ converges for $|\lambda| \leq 1$. Poincaré's memoir [33] of 1896, announced in note [32] of 18 February 1895, proved the convergence of Neumann's method when D *is simply connected and* S *has at each point a tangent plane and two curvature radii.* We do not give the details of this highly technical and today almost forgotten proof. The historical importance of [33] is, together with [29], to have inspired Ivar Fredholm in building his theory of *linear integral equations*. Indeed, the relations (19) give

$$\sum_{n=1}^{\infty} \lambda^n V_n(x) = -\lambda \int_S \frac{1}{2\pi} \sum_{n=1}^{\infty} \lambda^{n-1} V_{n-1}(s) \frac{\partial}{\partial n} \frac{1}{\|x-s\|}\, d\sigma,$$

i.e.

$$V(x) + \lambda \int_S \frac{1}{2\pi} \frac{\partial}{\partial n} \frac{1}{\|x-s\|} V(s)\, d\sigma + \int_S \frac{1}{2\pi} \frac{\partial}{\partial n} \frac{1}{\|x-s\|} \Phi(s)\, d\sigma = 0.$$

In other terms, V satisfies what is now called a *Fredholm integral equation of second kind* with kernel $\frac{1}{2\pi} \frac{\partial}{\partial n} \frac{1}{\|x-s\|}$. Paper [9] where Fredholm introduced his first study of an integral equation of second kind, entitled 'Sur une nouvelle méthode pour la résolution du problème de Dirichlet', starts as follows:

"In his profound researches (*Acta Mathematica* t. 20) on the convergence of Neumann's well-known method in potential theory, Mr. Poincaré has considered the Dirichlet problem as a special case of another problem, that he calls Neumann's problem. [...] Neumann has solved this problem in expanding the unknown function according to the power of a parameter λ. But it follows from Mr. Poincaré's researches that V is a meromorphic function. Hence it is clear that Neumann's series expansion cannot converge for all values of λ. But because we know that a meromorphic function can always be written as a quotient of two entire functions, I have found natural to search directly those entire functions."

Poincaré immediately grasped the importance of Fredholm's work on integral equations, generalized it in several ways and applied it to the theory of tides and to the diffraction of Hertzian waves. Half of Poincaré's lectures delivered in Göttingen in April 1909, upon Hilbert's invitation, deal with Fredholm integral equations and their applications [38].

Heat equation and telegraph equation

In the third section 'The laws of cooling' of his memoir [26], Poincaré considered the *non-stationary Fourier problem*, namely finding $V = V(t,x)$ such that $V(0,x) = V_0(x)$ $(x \in D)$ and

$$\begin{aligned} \frac{\partial V}{\partial t} &= a^2 \Delta V \quad \text{in }]0,\infty[\times D, \\ \frac{\partial V}{\partial n} + hV &= 0 \quad \text{on }]0,\infty[\times S, \end{aligned} \tag{20}$$

If the U_n are the fundamental functions of the stationary problem and k_n the corresponding characteristic numbers $(n = 1, 2, \ldots)$, one should be able to expand V_0 as $V_0 = \sum_{n=1}^{\infty} A_n U_n$, to deduce that, for all $t > 0$, $V(t,x) = \sum_{n=1}^{\infty} A_n e^{-a^2 k_n t} U_n(x)$. Observing that he "was not yet able to prove the possibility of the expansion in a general setting", Poincaré showed that the *averaged error*

$$S_0 := \int_D \left[V_0 - \sum_{p=0}^{n} A_p U_p \right]^2$$

is minimal when $A_p = \int_D V_0 U_p := J_p^0$ $(1 \leq p \leq n)$. For this choice of the coefficients A_p, if $J_p(t) = J_p^0 e^{-a^2 k_p t}$, Poincaré showed that one can take n large enough so that the averaged error made on the temperature at time t,

$$S(t) := \int_D \left[V(t,x) - \sum_{p=1}^{n} J_p(t) U_p(x) \right]^2 dx,$$

can be made as small as one wants. One recognizes in those considerations the beginning of the use of a *Hilbertian norm* instead of a uniform one in the study of a parabolic problem.

In an interesting note [27] in the *Comptes Rendus* of 26 December 1893, Poincaré gave the first general solution of the initial value problem for the *telegraph equation* on an indefinite line

$$\frac{\partial^2 V}{\partial t^2} + 2\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2}, \tag{21}$$

which describes the propagation of an electrical current in a conducting wire. This equation had been introduced in 1857 by Gustav Kirchhoff [12], starting from Weber's electromagnetic theory, and deduced from Maxwell's theory by Oliver Heaviside [11] in 1876. Those authors had only given special solutions of this equation. It is noticeable that Poincaré's motivation is not wire telegraphy, but to give a theoretical explanation to some experimental discrepancies in measuring the speed of propagation of electricity in wires. Letting $V = Ue^{-t}$, equation (21) becomes

$$\frac{\partial^2 U}{\partial t^2} = \frac{\partial^2 U}{\partial x^2} + U. \tag{22}$$

Given the initial conditions $U(0,x) = f(x)$ and $\frac{\partial U}{\partial t}(0,x) = f_1(x)$ in the form of Fourier integrals

$$f(x) = \int_{-\infty}^{+\infty} \theta(q) e^{iqx}\, dq, \quad f_1(x) = \int_{-\infty}^{+\infty} \theta_1(q) e^{iqx}\, dq,$$

searching the solution $U(x,t)$ in the form of a Fourier integral, Poincaré classically arrived to the expression

$$U(x,t) = \int_{-\infty}^{+\infty} e^{iqx} \left[\theta(q) \cos(t\sqrt{q^2-1}) + \theta_1(q) \frac{\sin(t\sqrt{q^2-1})}{\sqrt{q^2-1}} \right] dq$$

or, in complex form,

$$U(x,t) = \int_{-\infty}^{+\infty} \alpha(q) e^{i[qx + t\sqrt{q^2-1}]}\, dq + \int_{-\infty}^{+\infty} \beta(q) e^{i[qx - t\sqrt{q^2-1}]}\, dq$$

with

$$\alpha(q) = \frac{\theta(q)}{2} + \frac{\theta_1(q)}{2i\sqrt{q^2-1}}, \quad \beta(q) = \frac{\theta(q)}{2} - \frac{\theta_1(q)}{2i\sqrt{q^2-1}}.$$

For initial conditions $\theta \equiv 0$ and $\theta_1 \equiv 1$, contour integration allowed him to find the expression

$$U(x,t) = \begin{cases} 0 & \text{if } x > t \\ \Lambda(x,t) & \text{if } -t < x < t \\ 0 & \text{if } x < -t \end{cases}$$

with

$$\Lambda(x,t) := \frac{1}{2}\int_0^{2\pi} e^{ix\cos\varphi} e^{i\sin\varphi}\, d\varphi = \pi J_0(\sqrt{x^2-t^2}),$$

and J_0 is the Bessel function of index 0.

Poincaré then obtained a formula for the solution in the case where $f \equiv 0$ and $f_1(x) \neq 0$ if $b \leq x \leq a$, $f_1(x) = 0$ if $x < b$ or if $x > a$, and in the case where $f_1 \equiv 0$ and $f(x) \neq 0$ if $b \leq x \leq a$, $f(x) = 0$ if $x < b$ or if $x > a$. The combination of those two solutions gives the general solution when f and f_1 have compact support, and a detailed analysis of this solution allowed him to describe precisely the behaviour of an initial perturbation bounded in time and space: "One first sees that the head of the perturbation propagates with some speed, in such a way that, before this head, the perturbation is zero, in contrast to what happens in Fourier's heat theory, and according to the laws of propagation of light or sound through plane waves, deduced from the vibrating string equation. But there is, with respect to this last case, an important difference, because the perturbation, during its propagation, leaves behind a residue which is not zero [...] and can then trouble the observations."

So it is the diffusion of the wave in the wire which makes difficult the measure of its speed.

The study of the telegraph equation was considered with more details in [30–31, 35]. See [17] for a more complete description of Poincaré's contributions to the telegraph equation, and of his motivations.

A non-linear elliptic equation

The theory of Fuchsian functions leads to the study of solutions of the partial differential equation $\Delta u = ke^u$ $(k > 0)$, already considered by Joseph Liouville between 1847 and 1853, and by Émile Picard between 1890 and 1893. It is related to the theory of surfaces with constant negative curvature, closely linked to Lobatchevsky's geometry, and therefore to Fuchsian functions. Although Poincaré, in a memoir [37] of 1898, announced in a note [36] of 28 February of the same year, considered this equation on a Klein surface, we present his ideas, for simplicity, in the analogous more standard case of the homogeneous Neumann problem on a bounded planar domain D with boundary S.

Let θ be a positive smooth function and Φ a regular function on $\overline{D} := D \cup S$. The problem consists in finding a function u such that

$$\begin{aligned} &\Delta u = \theta(x)e^u - \Phi(x) \quad \text{in } D, \\ &\frac{\partial u}{\partial n} = 0 \quad \text{on } S. \end{aligned} \tag{23}$$

Starting from the fact that Neumann's problem for equation

$$\Delta u = \varphi(x) \quad \text{in } D \tag{24}$$

is solvable if and only if $\int_D \varphi = 0$, Poincaré first considered Neumann's problem for the *linear equation*

$$\Delta u = \eta u - \varphi(x) \quad \text{in } D, \tag{25}$$

where $\eta > 0$. To do so, he introduced the family of equations depending upon the parameter $\lambda > 0$,

$$\Delta u = \lambda\eta u - \varphi(x) - \lambda\psi(x) \quad \text{in } D, \tag{26}$$

with $\int_D \varphi = 0$, and showed that a unique solution of Neumann's problem for (26), of the form $u = \sum_{k=0}^{\infty} u_k\lambda^k$, exists if $2\beta\lambda < 1$, with $\beta > 0$ a constant related to problem (24). Assuming then that Neumann's problem for equation

$$\Delta u = \eta u - \Phi(x) \quad \text{in } D$$

has a unique solution for every Φ, Poincaré showed that the same problem for

$$\Delta u = (\eta + \lambda\eta')u - \Phi(x) \quad \text{in } D \tag{27}$$

with $\eta' > 0$ has a unique solution, of the form $u = \sum_{k=0}^{\infty} u_k\lambda^k$, when $\lambda < \eta/\eta'$.

To deal with (25), Poincaré introduced p positive numbers $\lambda_1, \ldots, \lambda_p$ such that $\sum_{k=1}^{p} \lambda_k = 1$, and wrote $\varphi(x) = \varphi_1(x) + \lambda_1\psi_1(x)$, with $\int_D \varphi_1 = 0$. From the result on (26), Neumann's problem for equation

$$\Delta u = \lambda_1\eta u - \varphi_1(x) - \lambda_1\psi_1(x) \quad \text{in } D$$

has a unique solution if $\lambda_1 < 1/2\beta$. From the result on (27), the same is true for equation

$$\Delta u = \lambda_1\eta u + \lambda_2\eta u - \varphi(x) \quad \text{in } D$$

when $\lambda_2 < \lambda_1$, and for equation

$$\Delta u = (\lambda_1 + \lambda_2)\eta u + \lambda_3\eta u - \varphi(x) \quad \text{in } D$$

when $\lambda_3 < \lambda_1 + \lambda_2$. Continuing in the same way, Neumann's problem for equation

$$\Delta u = (\lambda_1 + \cdots + \lambda_{p-1})\eta u + \lambda_p\eta u - \varphi(x) \quad \text{in } D,$$

i.e. for equation (25), has a unique solution if $\lambda_p < \lambda_1 + \cdots + \lambda_{p-1}$. The conditions on the λ_k can all be fulfilled by taking p sufficiently large. The underlying idea to this proof is clearly a *continuation method*.

Returning to the *non-linear problem* (23), integrating both members of the equation over D, using the Green formula and the boundary condition, Poincaré showed that $\int_D \Phi > 0$ was a *necessary* condition for the existence of a solution to (23). If U denotes a solution of Neumann's problem for the linear equation

$$\Delta u = -\Phi(x) + \frac{1}{|D|}\int_D \Phi \quad \text{in } D,$$

the change of variable $u = U + v$ leads to the equivalent Neumann problem for equation

$$\Delta v = \theta(x)e^{U(x)}e^{v} - \frac{1}{|D|}\int_D \Phi \quad \text{in } D,$$

which, together with the necessary condition, shows that it suffices to consider the case of a positive Φ.

Poincaré noticed that a first idea to prove the existence of a solution consists in using *Dirichlet's principle*, i.e. finding a function u minimizing the energy integral

$$\int_D \left[\frac{\|\nabla u(x)\|^2}{2} + \theta e^{u(x)} - \Phi(x)u(x)\right] dx,$$

bounded from below when Φ is positive, but the existence of a minimum is not guaranteed. Hence Poincaré returned to the *continuation method* already used for (25). Assuming that problem

$$\Delta u = \theta(x)e^{u} - \varphi(x) \quad \text{in } D, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } S$$

has a unique solution u_0, Poincaré introduced the family of equations

$$\Delta u = \theta(x)e^{u} - \varphi(x) - \lambda\psi(x) \quad \text{in} \quad D, \tag{28}$$

and searched a solution for Neumann's problem in the form $u = \sum_{k=0}^{\infty} u_k\lambda^k$. Letting

$$e^u = 1 + u + \sum_{k=2}^{\infty} w_k\lambda^k,$$

he showed that $w_k = w_k(u_1, \ldots, u_{k-1})$ and that $\max_{T\cup S}|u_k| \le \max_{\overline{D}}|w_k|$ for every $k \ge 2$. The convergence of $\sum_{k=1}^{\infty} u_k\lambda^k$, follows, as well as the existence of a unique solution to (28) if

$$|\lambda| < (\log 4 - 1)\frac{\max_{\overline{D}}(|\varphi|/\theta)}{\max_{\overline{D}}(|\psi|/\theta)}.$$

But, for any $\alpha > 0$, the problem

$$\Delta u = \theta(x)e^{u} - \alpha\theta(x) \quad \text{in } D, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } S$$

has the unique solution $u(x) = \log\alpha$. Hence, if ψ is positive, Neumann's problem for equation

$$\Delta u = \theta(x)e^{u} - \alpha\theta(x) - \lambda\psi(x) \quad \text{in } D$$

has a unique solution for $0 \le \lambda < \alpha(\log 4 - 1)/\max_{\overline{D}}(\psi/\theta)$, the same is true for

$$\Delta u = \theta(x)e^{u} - \alpha\theta(x) - \lambda\psi(x) - \mu\psi(x) \quad \text{in } D,$$
$$\frac{\partial u}{\partial n} = 0 \quad \text{on } S$$

if

$$0 \le \mu < (\log 4 - 1)\frac{\max_{\overline{D}}[\alpha + (\lambda\psi/\theta)]}{\max_{\overline{D}}(\psi/\theta)},$$

and hence in particular if $0 \le \mu < \lambda < \alpha(\log 4 - 1)/\max_{\overline{D}}(\psi/\theta)$. Continuing in this way, problem (28) has a unique solution for any $0 \le \lambda < n\alpha(\log 4 - 1)/\max_{\overline{D}}(\psi/\theta)$, and any $n \ge 1$. Taking now $\psi = \Phi - \alpha\theta$, with $0 < \alpha < \min_{D\cup S}(\Phi/\theta)$ in order that ψ be positive, and taking n such that $n\alpha(\log 4 - 1)/\max_{\overline{D}}[(\Phi - \alpha\theta)/\theta] > 1$, Neumann's problem for equation

$$\Delta u = \theta(x)e^{u} - \alpha\theta(x) - \lambda[\Phi(x) - \alpha\theta(x)] \quad \text{in } D$$

has a unique solution for $\lambda = 1$, which is the wanted result.

Notice that a very similar approach had been used and very clearly presented, for a Dirichlet problem associated to semi-linear elliptic equations, in the PhD thesis of Édouard Le Roy [13–14] (dedicated to Poincaré, who also wrote the report). Despite of the proximity of publication dates, neither author quoted the other one. As mentioned in the Introduction of [37], Poincaré had previously, like Félix Klein, used a continuation method, in a finite dimensional setting, in the theory of Fuchsian functions. The method and results of Le Roy and of Poincaré are briefly described by Arnold Sommerfeld in [42]. His short description starts with: "Finally, to deal with the equations $\Delta u + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + c\frac{\partial u}{\partial z} = fu$ and $\Delta u + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + c\frac{\partial u}{\partial z} = F(x, y, z, u)$, Ed. Le Roy has introduced the following method of analytic continuation..." and ends as follows: "H. Poincaré also uses the same ideas in his researches referred in Nr. 12 [$\Delta u = ke^u$]." Consequently, the introduction of a continuation method to study semi-linear elliptic boundary value problems, generally attributed to Sergei N. Bernstein's paper [4] of 1906, must be traced to Le Roy and Poincaré's contributions of 1898, which are not quoted in [4], despite of the very close relations of Bernstein with the French mathematical community. For more details on Le Roy's and Poincaré's contributions, and the development of the continuation method, see [18].

Partial differential equations in Poincaré's monographs

When he occupied the chair of mathematical physics and probabilities at the University of Paris, Poincaré changed almost every semester the topics of his lectures. Most of them were then written down by one or two (bright) students and revised by Poincaré before publication. Although partial differential equations are present in other ones, we only describe the most significant ones.

The telegraph equation is discussed in the monograph [30], describing Poincaré's lectures on electrical oscillations during the first semester of the academic year 1892–1893. It is edited by Charles Maurain (1871–1967), a future well-known geophysicist. The telegraph equation is introduced and discussed on pp. 182–189 of a 'Complement to Chapter IV' starting as follows: "Since this course has been given, several experiments have been done and I will be forced to add from time to time several supplementary lines to inform the reader about the present state of science; in this way, I must mention the nice experiments of M. Blondlot on the propagation of electricity."

After describing the experiments of Hippolyte Fizeau (1819–1896) and Eugène Gounelle (1821–1864) of 1850 in order to measure the speed of propagation of electricity in metallic wires, Poincaré mentioned that: "Everything happens as if the perturbation shaded off in such a way to occupy more space on the wire at the arrival than at the departure. This phenomenon, placed beyond any doubt by the experiments of M. Fizeau, has been called by this physicist the diffusion of the current. [...] It is likely that this diffusion is due to the Ohmic resistance that we have neglected up to now."

Poincaré then showed how the equations of propagation of electricity are modified by the presence of this resistance, and arrived, for the variations of the electrical potential in a wire transmitting an electrical perturbation, to the equation

$$A\frac{\partial^2 V}{\partial t^2} + 2B\frac{\partial V}{\partial t} = C\frac{\partial^2 V}{\partial x^2},$$

which is known under the name of *telegraph equation* (*équation des télégraphistes* in French).

Poincaré then solved this equation in a way entirely similar to the one given in [27].

During the first semester of the academic year 1893–1894, Poincaré devoted his lectures to the analytical theory of the propagation of heat. They were written down by Rouyer and René Baire (1874–1932). The book starts as follows:

"Fourier's heat theory is one of the first examples of the application of analysis to physics. [...] The results he has obtained are surely interesting by themselves, but what is still more interesting is the method he has used to reach them, which always will remain a model for those who will want to cultivate a branch of mathematical physics. I will add that Fourier's book has a fundamental importance in the history of mathematics, and that pure analysis maybe owes him still more than applied analysis."

After discussing the various types of propagation of heat, Poincaré established Fourier's heat equation and gave it in various systems of coordinates. The case of an infinite rectangular solid was then treated by separation of variables and Fourier series (giving Poincaré an opportunity to recall Dirichlet's and Abel's contributions). Fourier series are then used to solve Fourier's equation on a closed wire. The case of an infinite wire led Poincaré to Fourier's integral whose main properties are established. Follows then an interesting chapter on 'linear equations analogous to that of heat', where the vibrating string and telegraph equations are solved using the method of Fourier integrals, and their solutions compared to those of heat equation in the case of an initial perturbation with compact support.

The heat equation on an infinite wire or solid is then solved by Laplace method, and the justification of Fourier's formal results on the cooling of a sphere, led Poincaré to a description of Cauchy's theory on asymptotic values of some integrals. The case of the cooling of an arbitrary body D with boundary S is reduced, by separation of time and space variables, to the problem of finding a non-trivial function $U(x, y, z)$ and a number k such that

$$\Delta U + kU = 0 \quad \text{in } D, \quad \frac{\partial U}{\partial n} + hU = 0 \quad \text{on } S.$$

For this problem, "losing in rigour what we will gain in generality", Poincaré reproduced his results of the memoir [26] giving the corresponding eigenvalues and eigenfunctions by an extension of Dirichlet's principle, and only quoted the second memoir [29] where the rigorous proof of existence of eigenvalues and eigenfunctions is given. The remaining part of the book is devoted to the use of special functions in solving various problems of analytical heat theory.

Poincaré's lectures of the first semester of the academic year 1894–1895, devoted to the theory of Newtonian potential, were written down by Édouard Le Roy and Georges Vincent. The first part of the book gives a rather standard presentation of the computation of the various integrals expressing the Newtonian potential for volumes, surfaces and lines. As a curiosity (called to my attention by Michel Willem), one should mention Poincaré's remark that many of the used results on particular integrals of potential theory are a special case of the following theorem on integrals, stated on p. 121:

Theorem. *Consider the integral*

$$\int_{(S)} f(x, y, z)\, dx\, dy$$

extended to some domain S. Assume that the following conditions hold:

1. *The curve C bounding S does not depend upon z.*
2. *One has, at any point of S,*

$$f(x, y, z) < \varphi(x, y)$$

where φ is a positive function.

3. *The integral $\int \varphi(x, y)\, dx\, dy$, extended to the domain S, exists.*
4. *Finally one has*

$$\lim_{z\to 0} f(x, y, z) = f(x, y, 0)$$

for any fixed x, y.

In those conditions, one has the relation

$$\lim_{z=0} \int_{(S)} f(x, y, z)\, dx\, dy = \int_{(S)} f(x, y, 0)\, dx\, dy.$$

The reader has already noticed the similarity of the statement with that of Lebesgue's dominated convergence theorem, even if the background here is, of course, not Lebesgue integration theory, but some improper integrals. Poincaré then described the properties of harmonic functions (mean value and maximum principles), and the use of Green's functions in solving the Dirichlet problem, followed by the special case of the circle and the sphere, and the double layer potential.

The last hundred pages of the book are more closely related to Poincaré's recent researches in potential theory, giving a presentation of his results of [26] on the sweeping out method (Chapter VII), Neumann's method on convex sets (Chapter VIII), and his extension of Neumann's method to simply connected domains given in [33] (Chapter IX).

This short description suffices to show that, in his lectures, Poincaré not only gave a masterly presentation of the classical theory of partial differential equations of mathematical physics and their evolution, but did not hesitate to present his most recent results in this area. It is hard to find a better illustration of the necessary and wonderful interaction between research and teaching at the university level.

Conclusion

Poincaré's contributions to partial differential equations would have been amply sufficient to place him among the greatest mathematicians of the end of the nineteenth century and the beginning of the twentieth century. But Poincaré is also the father of automorphic functions, dynamical systems and chaos, algebraic topology and modern celestial mechanics, to quote only his main achievements [15]. He is one of the greatest mathematicians of history, and, in addition, his important contributions to physics and to the philosophy of science place him at a high level in those disciplines.

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