

Problemen

| Problem Section

This Problem Section is open to everyone; everybody is encouraged to send in solutions and propose problems. Group contributions are welcome.

For each problem, the most elegant correct solution will be rewarded with a book token worth 20 Euro. At times there will be a Star Problem, to which the proposer does not know any solution.

For the first correct solution sent in within one year there is a prize of 100 Euro.

When proposing a problem, please either include a complete solution or indicate that it is intended as a Star Problem. Electronic submissions of problems and solutions are preferred (problems@nieuwarchief.nl).

The deadline for solutions to the problems in this edition is June 1st, 2012.

Problem A (proposed by Hendrik Lenstra)

Let m and n be coprime positive integers. Let Γ be the graph that has the disjoint union $\mathbb{Z}/n\mathbb{Z} \sqcup \mathbb{Z}/m\mathbb{Z}$ as vertex set and that has for every $1 \leq i \leq m+n-1$ an edge connecting $i \bmod n$ and $i \bmod m$. Show that Γ is a tree.

Problem B (proposed by Gabriele Dalla Torre)

Is it possible to partition a non-empty open interval in closed intervals of positive length? Let Δ be a triangle (including its interior) and let $P \in \Delta$ be an interior point. Is it possible to partition $\Delta - \{P\}$ in closed line segments of positive length?

Problem C (proposed by Tejaswi Navilarekallu)

A move on a pair (a, b) of integers consists of replacing it with either $(a+b, b)$ or $(a, a+b)$. Show that starting from any pair of coprime positive integers one can obtain a pair of squares in finitely many moves.

Edition 2011-3 We received solutions from Pieter de Groen (Brussel), Alex Heinis (Hoofddorp), Tejaswi Navilarekallu (Amsterdam), Hendrik Reuvers (Maastricht) and Albert Stadler (Herrliberg).

Problem 2011-3/A Fix a point P in the interior of a face of a regular tetrahedron Δ . Show that Δ can be partitioned in four congruent convex polyhedra such that P is a vertex of one of them.

Solution We received solutions from Pieter de Groen, Alex Heinis, Tejaswi Navilarekallu and Hendrik Reuvers. The book token goes to Hendrik Reuvers.

Let A, B, C and D be the vertices of Δ . Define $\sigma_1 = (AB)(CD)$ to be the rotation of Δ that interchanges A with B and C with D . Similarly write $\sigma_2 = (AC)(BD)$ and $\sigma_3 = (AD)(BC)$. Then $V_4 = \{\text{id}, \sigma_1, \sigma_2, \sigma_3\}$ is a subgroup of the symmetry group of Δ .

Let P_a, P_b, P_c and P_d be the images of P under V_4 in the faces BCD, CDA, DAB and ABC , respectively. (For example, if P lies in face ABC , then $P = P_d$.) Let Z be the orthocentre of Δ . Define Q_{ab} to be the intersection point of AB with the plane through Z, P_c and P_d , and analogously define $Q_{ac}, Q_{ad}, Q_{bc}, Q_{bd}$ and Q_{cd} .

Now the four polyhedra

$$AP_bP_cP_dQ_{ab}Q_{ac}Q_{ad}Z,$$

$$BP_cP_dP_aQ_{bc}Q_{bd}Q_{ab}Z,$$

$$CP_dP_aP_bQ_{cd}Q_{ac}Q_{bc}Z,$$

$$DP_aP_bP_cQ_{ad}Q_{bd}Q_{cd}Z$$

cover Δ and are congruent, as they are mapped onto each other by the symmetries in V_4 . The polyhedra are convex since each of them is the intersection of six half-spaces, three of which are defined by a plane containing a face of Δ and three by a plane through Z .

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Oplossingen

| Solutions

Problem 2011-3/B Let n be a positive integer. Show that 3^n divides the numerator of

$$\sum_{k=1}^n \frac{4k-1}{2k(2k-1)} 9^k.$$

Solution We received solutions from Alex Heinis, Tejaswi Navilarekallu and Albert Stadler. The book token goes to Albert Stadler.

We use the fact that if $q \in \mathbb{Q}$ has denominator not divisible by 3, then q can be reduced modulo 3^k for all positive integers k .

Since $2 \log(1-3x) = \log(1-6x+9x^2)$ as complex functions, we have the following identity of power series:

$$2 \sum_{k=1}^{\infty} \frac{3^k x^k}{k} = \sum_{k=1}^{\infty} \frac{3^k (2x-3x^2)^k}{k}. \tag{1}$$

The coefficients (in x) of these power series are rational numbers. Since for all $k \geq 1$ we have $k \leq 3^k$, it follows that the denominator of $3^k/k$ is not divisible by 3, and hence that none of the coefficients of these power series have denominator divisible by 3.

In particular, we can reduce these coefficients modulo 3^n . For all $k \geq 2n$ we have that 3^n divides the numerator of $3^k/k$, so we obtain from (1) the congruence

$$\sum_{k=1}^{2n} \frac{2 \cdot 3^k x^k}{k} \equiv \sum_{k=1}^{2n} \frac{3^k (2x-3x^2)^k}{k} \pmod{3^n}.$$

By substituting $x = 1$, we obtain the following identity modulo 3^n :

$$\begin{aligned} 0 &\equiv \sum_{k=1}^{2n} \frac{2 \cdot 3^k - (-3)^k}{k} \\ &\equiv \sum_{k=1}^n \frac{3^{2k}}{2k-1} + \sum_{k=1}^n \frac{3^{2k}}{2k} \\ &\equiv \sum_{k=1}^n \left(\frac{1}{2k-1} + \frac{1}{2k} \right) 9^k \\ &\equiv \sum_{k=1}^n \frac{4k-1}{2k(2k-1)} 9^k. \end{aligned}$$

Hence 3^n divides the numerator of

$$\sum_{k=1}^n \frac{4k-1}{2k(2k-1)} 9^k,$$

as desired.

One can also prove identity (2) for $x = 1$ by looking at the so-called *3-adic logarithms* of -2 and 4 on the *3-adic integers*, see for example the book *p-adic Numbers: An Introduction* by F.Q. Gouvea.

Problem 2011-3/C Let $n > 1$ be an integer. Show that there are no non-linear complex polynomials $f(X)$ such that

$$f^n(X) - X = (f \circ f \circ \dots \circ f)(X) - X$$

is divisible by $(f(X) - X)^2$.

Solution We received a solution from Alex Heinis, who wins the book token. Assume f satisfies the condition. Put $g(X) := f(X) - X$.

Opllossingen

| Solutions

We claim that

$$f^n(X) - X \equiv g(X) (1 + f'(X) + \dots + f'(X)^{n-1})$$

modulo $g(X)^2$ for all $n \geq 1$.

Clearly the claim holds for $n = 1$. Assume it holds for $n = N - 1$. Using that for all polynomials a we have

$$f(X + g(X)a) \equiv f(X) + g(X)af'(X) \pmod{g(X)^2},$$

we find

$$\begin{aligned} f^N(X) - X &\equiv f(f^{N-1}(X) - X + X) - X \\ &\equiv f(X + g(X)(1 + f'(X) + \dots + f'(X)^{N-2})) - X \\ &\equiv g(X) + f'(X)g(X)(1 + f'(X) + \dots + f'(X)^{N-2}) \\ &\equiv g(X)(1 + f'(X) + \dots + f'(X)^{N-1}), \end{aligned}$$

which proves the claim.

Since $g(X)^2$ divides $f^n(X) - X$ we find that $g(X)$ divides

$$1 + f'(X) + \dots + f'(X)^{n-1} = \frac{f'(X)^n - 1}{f'(X) - 1}.$$

Let $x \in \mathbb{C}$ be a root of $g(X)$. From the above equation we find that $f'(x)^n = 1$. We claim that $f'(x) \neq 1$, and in particular that x is a simple root of $g(X)$. To see this, assume that $f'(x) = 1$. Let k be the multiplicity of x as a root of $f'(X) - 1$. Then x must be a root of multiplicity $k + 1$ of $f'(X)^n - 1$, and hence a k -fold root of its derivative $n f'(X)^{n-1} f''(X)$. But since $f'(x) = 1$ it follows that the multiplicity of x as a root of $f''(X)$ is k , a contradiction.

Now assume that f is not linear. Clearly f cannot be constant, so the degree of f is at least 2, and hence also the degree of g is at least 2. But then the residue theorem gives

$$0 = \sum_{g(x)=0} \frac{1}{g'(x)} = \sum_{g(x)=0} \frac{1}{f'(x) - 1},$$

where the sums range over all roots x of $g(X)$. Since $f'(x)^n = 1$ for all such x , we have that all the $f'(x)$ lie on the unit circle. In particular, the real part of the right-hand side is strictly negative, a contradiction.

