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History

Mathematics and its worldwide history

Perhaps more than any other science, mathematics contains its own history, both as a field of study and as a professional discipline. Last year the Royal Dutch Mathematical Society (KWG) has selected the field of history of mathematics for the Brouwer Prize. During this tri-annual event Kim Plofker of the Union College in Schenectady, USA, received the Brouwer Medal. She is an expert in the history of science in Antiquity and in the Middle Ages, and in Sanskrit. This article is based on the Brouwer Lecture that she delivered on 14 April 2011 during the annual Dutch Mathematical Congress. She discusses changing aims and current priorities in the history of mathematics, with special reference to comparisons between 'western' and 'non-western' traditions of mathematical knowledge. Among her examples she offers some remarkable results from late-medieval Indian mathematics that illustrate the creative tension between mathematical experiment and proof.

Mathematics as practiced in the currently dominant tradition extending roughly from Euclid to contemporary research has seldom had to discard any of its earlier discoveries to make room for later ones, as physics has done with its theory of the four elements or medicine with its theory of the four humors. The arguments of earlier mathematicians working in this tradition may come to seem outdated or insufficiently rigorous, but are rarely rejected as actually false. (For more detailed historiographic discussions of evolution and 'revolution' in mathematical knowledge and methodology, see [11, 24].) Consequently, the modern study of mathematics retains closer links to its own history than most other quantitative sciences do. A specific problem may remain open, in more or less its original formulation, for hundreds of years before attaining a definitive solution (as in the famous examples of Fermat's Last Theorem and the Kepler conjecture), or may still await solution after more than a millennium (as do several conjectures about perfect numbers).

History as a mathematical subject

While it may be somewhat hyperbolic to claim that mathematics as an intellectual discipline actually contains its own history, it is undeniable that mathematics as a professional field for the most part contains its own historians. This sharply contrasts with, say, political or economic history or history of art, most of whose researchers have little or no formal training or professional experience as legislators or economists or artists. Even a field like history of science, requiring in many areas a considerable amount of specialized technical knowledge, is professionally dominated by historians rather than by scientists. Research on the history of mathematics, on the other hand, is and has always been produced chiefly by people trained and/or professionally classified as mathematicians: simply put, few scholars outside mathematics have the requisite interest or technical background in the subject matter. This is especially true for the study of historical developments in advanced and recent mathematics, but even topics requiring no more than basic calculus or pre-calculus knowledge are often shunned by non-mathematicians. Analyses of the cultural and intellectual isolation of mathematics from other subjects are still often rooted in C.P. Snow's 'two cultures' thesis [29] or Sarton's earlier articulation of the 'essential difference' of mathematics [28]; for a more recent survey of some of the issues involved, see [24].

History of mathematics research is thus a body of knowledge largely created by mathematicians for mathematicians, reflecting mostly what mathematicians themselves have seen in mathematics. The remainder of this section sketches some of the reasons why they have chosen to do so: namely, what are the goals and incentives that have led mathematicians to adopt mathematics history for all practical purposes as a subfield of mathematics?

The rediscovery of earlier mathematics

We may take it for granted that throughout the existence of mathematics as a literate discipline, many mathematicians have felt some degree of curiosity about its evolution and the origins of specific technical concepts and conventions. But during most of the western mathematical tradition, they did not systematically investigate earlier texts and record their reconstructions of how mathematical knowledge was shaped. (One notable exception, the classical Greek historiography of mathematics that flourished over two millennia ago, will not be discussed in detail here. A recent survey of classical doxographic and historical studies of science, including much material on mathematics and referencing many earlier studies, is [36].) When they did undertake such investigations, two motives in particular seem to have impelled them.

The first of these concerned the 'applied' side (stretching the modern concept of 'applied mathematics' to include exact sciences such as astronomy, which historically overlapped with mathematics as a disciplinary and professional category). Historical sources in this category, such as mathematical astronomy texts, were sometimes associated with useful records of observational data. For 'pure' mathematics researchers, on the other hand, ancient texts might contain lost esoteric or advanced theoretical knowledge.

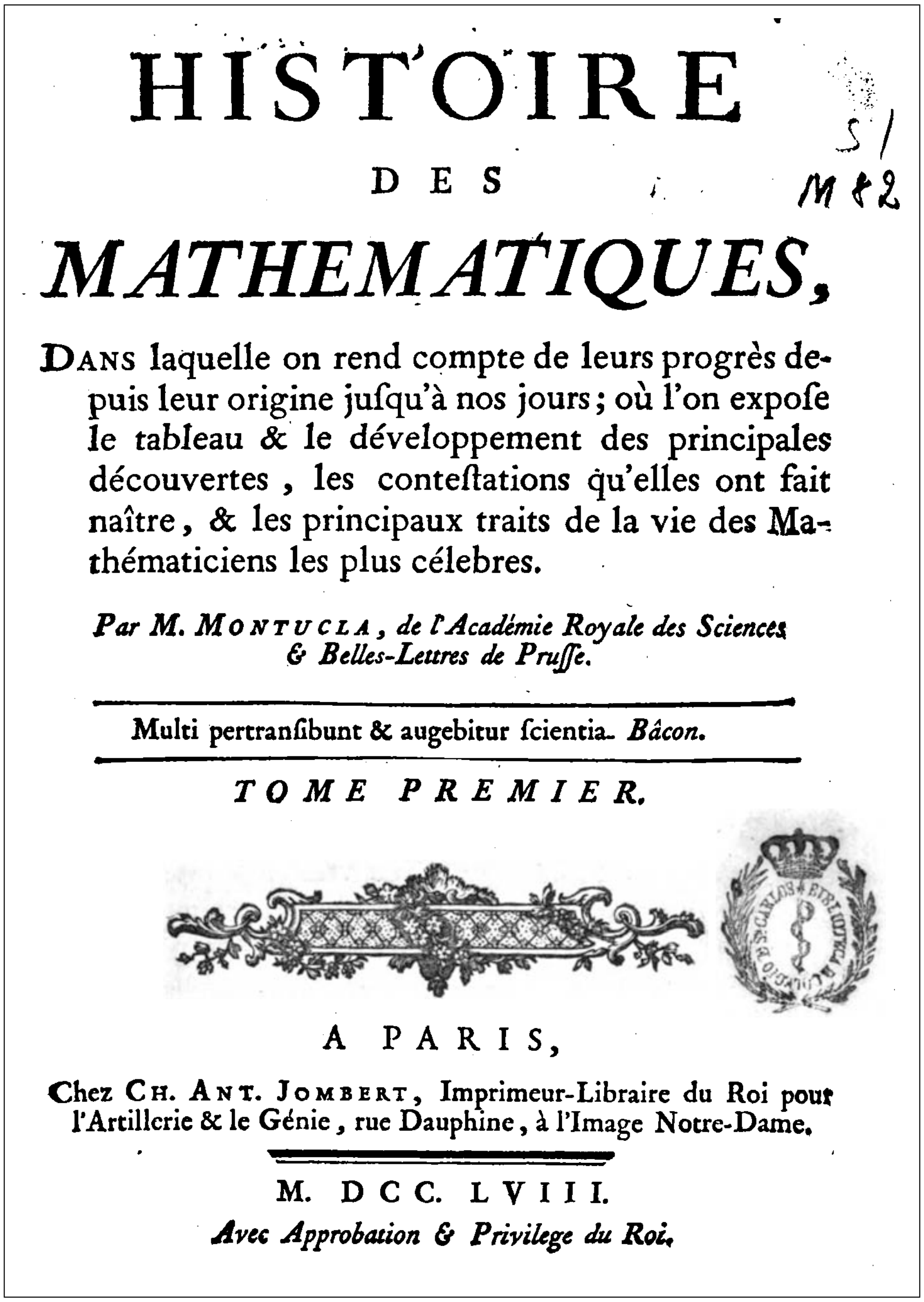
A notable example of the first quest is the examination of Babylonian mathematical astronomy texts from before the common era, in search of recorded data that might lengthen the observational baseline over which models in celestial mechanics might be tested. Such efforts had begun already in classical antiquity, as the work of Hipparchus and Ptolemy shows [35, pp. 190–216]. Ancient data is still used in some contemporary research, e.g., [32]. The historical background of this 'applied historical astronomy' is discussed in [31].

Examples of the latter kind can be seen in the Renaissance rediscovery of classical Greek mathematics, which appeared to many practitioners far more sophisticated than more recent work. Understanding the ideas of 'the ancients' was thus perceived as more worthwhile than merely following in the track of their own immediate predecessors, 'the moderns' — an opinion often strengthened by xenophobic suspicion of 'Oriental' influences in post-Greek thought. For instance, François Viète, one of the first developers of modern symbolic notation in algebra, was persuaded that he was merely rediscovering and re-formatting a solution technique that must have been familiar to ancients like Diophantus but was 'debased and polluted' by 'barbarians':

"...The art that I bring forth is new, but ultimately so ancient, and debased and polluted by barbarians... But underneath that which they extol and call the 'great art', Algebra or Almucabala, all mathematicians recognize the unmatched gold lies hidden; however, they have hardly found it... The way of seeking the truth in mathematics is one that Plato is said to have first found, named by Theon Analysis..."

("...Ecce ars quam profero nova est, aut demum ita vetusta, & à barbaris defaedata & conspurcata... At sub suâ, quam predicabant, & magnam artem vocabant, Algebrâ vel Almucabalâ, incomparandum latere aurum omnes adgnoscebant Mathematici, inveniebant verò minimè... Est veritatis inquirendae via quaedam in Mathematicis, quam Plato primus invenisse dicitur, à Theone nominata Analysis..." [33, pp. 2–4])

While the ancient texts did not reveal all the secrets that their editors hoped for, they



HISTOIRE

DES

MATHEMATIQUES,

DANS laquelle on rend compte de leurs progrès depuis leur origine juſqu'à nos jours; où l'on expoſe le tableau & le développement des principales découvertes, les conteſtations qu'elles ont fait naître, & les principaux traits de la vie des Mathématiciens les plus célebres.

Par M. MONTUCLA, de l'Académie Royale des Sciences & Belles-Lettres de Pruſſe.

Multi pertranſibunt & augebitur ſcientia. *Bâcon.*

TOME PREMIER.

A PARIS,

Chez CH. ANT. JOMBERT, Imprimeur-Libraire du Roi pour l'Artillerie & le Génie, rue Dauphine, à l'Image Notre-Dame.

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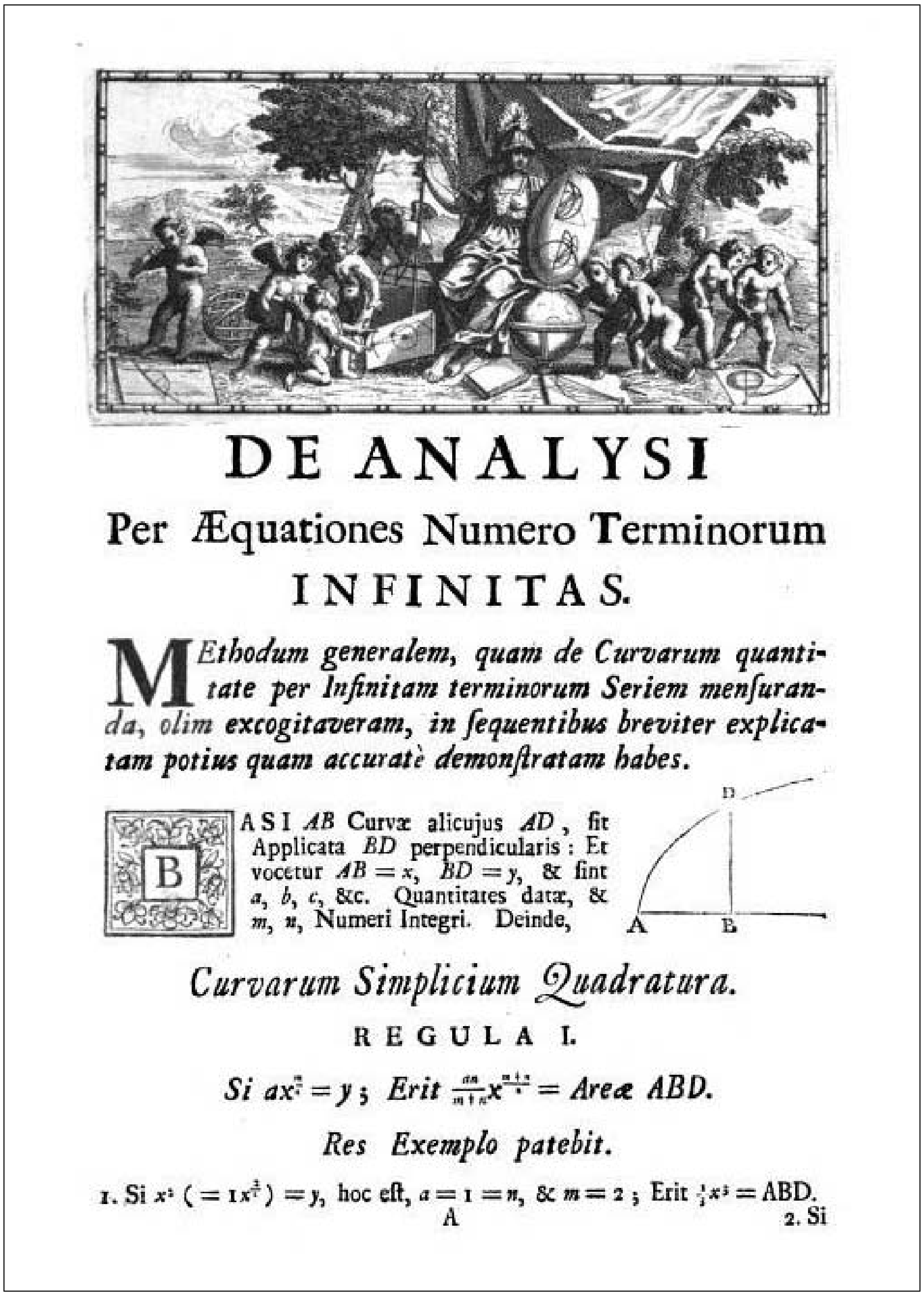
Figure 1 The title page of Montucla's seminal history



DE ANALYSI

Per Æquationes Numero Terminorum

INFINITAS.

Methodum generalem, quam de Curvarum quantitate per Infinitam terminorum Seriem menſurandа, olim excogitaveram, in ſequentibus breviter explicatam potius quam accuratè demonſtratam habes.

BASI *AB* Curvæ alicujus *AD*, ſit Applicata *BD* perpendicularis: Et vocetur $AB = x$, $BD = y$, & ſint a, b, c, &c. Quantitates datæ, & m, n, Numeri Integri. Deinde,

Curvarum Simplicium Quadratura.

REGULA I.

Si $ax^{\frac{m}{n}} = y$; *Erit* $\frac{an}{m+n}x^{\frac{m+n}{n}} =$ *Areæ ABD.*

Res Exemplo patebit.

1. Si $x^2 \ (= 1x^{\frac{2}{1}}) = y$, hoc eſt, $a = 1 = n$, & $m = 2$; Erit $\frac{1}{3}x^3 =$ ABD.
A
2. Si

Figure 2 The title page of Newon's *De analysi*

did inspire not only new research in mathematics but also an abiding interest in its chronological development.

Emergence in the eighteenth century

The history of mathematics as a subject distinct from mathematical research itself grew out of early modern mathematicians' efforts to piece together the widely scattered contributions to their rapidly expanding discipline. Like the field of physics in the mid-twentieth century or biology a few decades later, mathematics in the eighteenth century was adjusting to major breakthroughs that substantially increased its prominence both among the sciences and in society as a whole.

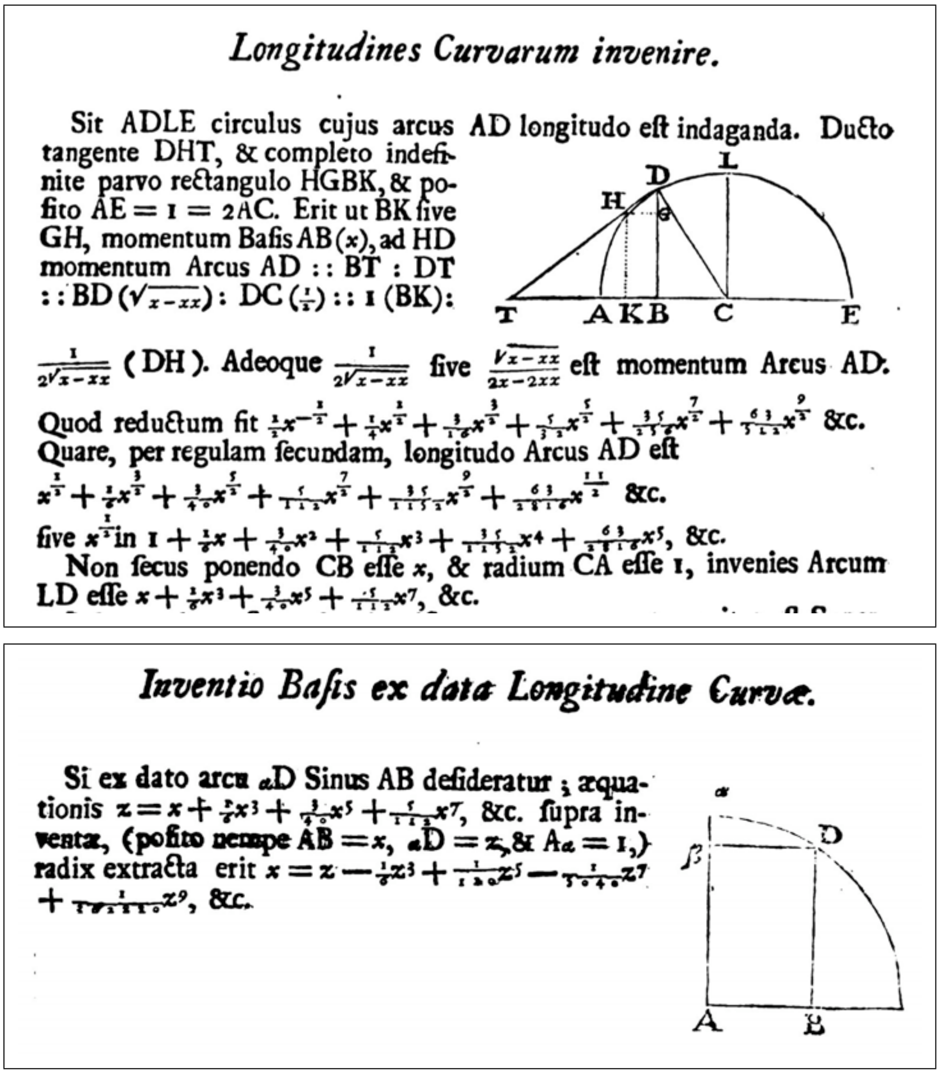
Trying to devise classification schemes that would systematically incorporate all the proliferating branches of their subject and link them to contemporary grand projects for tracing the history of human thought as a whole, researchers turned their attention to the underlying structure of mathematics and sought to explain how it had been shaped [19, pp. 6–11]. Both the triumphalism inspired by technical advancement and the focus on identifying its historical genesis are evident in what is widely considered the first independent work on history of mathematics, Jean-Etienne Montucla's 1758 *Histoire des mathématiques* or *History of Mathematics* (Figure 1), whose title page proclaims that therein

"... its progress from its origin to our own day is accounted for [and] the outline and development of the principal discoveries, the disputes it has given rise to, and the principal features of the lives of the most celebrated mathematicians are explicated." [16]

Montucla's book and many of its successors were primarily concerned with recording the technical details of the discoveries it chronicled, and establishing who was entitled to claim the credit for them — a natural preoccupation in light of some of the bitter priority disputes that convulsed seventeenth- and eighteenth-century mathematics. But as the nineteenth and twentieth centuries brought increased professionalization to this new field, with journals, university positions, conferences and congresses embracing or exclusively devoted to history of mathematics, more researchers began to grapple with the deeper challenges of being historians as well as mathematicians [7, *passim*].

New discoveries and corrections

Mathematicians and mathematics teachers accustomed to thinking of their work in terms of rigorous deductions concerning indisputable objective knowledge could sometimes be over-confident in their approach to historical inferences. The creation and subsequent debunking of various mathematical myths underlined the need for caution. Perhaps the best-known instances of such historical fallacies centered around the debate over the origin and transmission of the 'Arabic' or decimal place-value numerals. Although a few late medieval European texts, following statements in Arabic works, had already accurately described these 'ten ciphers' as borrowed by mathematicians in the Islamic world from Indian sources [20], some later researchers fell victim to more speculative hypotheses. The prolific nineteenth-century historian of mathematics Moritz Cantor, for example, suggested that the original versions of the decimal place-value numerals were the work of Greek Pythagoreans, supplemented by a zero borrowed from Indian arithmetic and conveyed into the Latin tradition via the work of Boethius in the sixth century. (See [7, p. 388; 3, pp. 231–250]. Historians of mathematics in the early twentieth century devoted considerable effort to unravelling various discredited speculations of this sort: [30; 2, pp. 64–68].)


Longitudines Curvarum invenire.

Sit ADLE circulus cujus arcus AD longitudo est indaganda. Ducto tangente DHT, & completo indefinite parvo rectangulo HGBK, & posito AE = 1 = 2AC. Erit ut BK sive GH, momentum Basis AB (x), ad HD momentum Arcus AD :: BT : DT :: BD ($\sqrt{x-xx}$) : DC ($\frac{1}{2}$) :: 1 (BK) : $\frac{1}{2\sqrt{x-xx}}$ (DH). Adeoque $\frac{1}{2\sqrt{x-xx}}$ sive $\frac{\sqrt{x-xx}}{2x-2xx}$ est momentum Arcus AD.

Quod reductum fit $\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{4}x^{\frac{1}{2}} + \frac{3}{16}x^{\frac{3}{2}} + \frac{5}{32}x^{\frac{5}{2}} + \frac{35}{256}x^{\frac{7}{2}} + \frac{63}{512}x^{\frac{9}{2}}$ &c.
Quare, per regulam secundam, longitudo Arcus AD est
$x^{\frac{1}{2}} + \frac{1}{6}x^{\frac{3}{2}} + \frac{3}{40}x^{\frac{5}{2}} + \frac{5}{112}x^{\frac{7}{2}} + \frac{35}{1152}x^{\frac{9}{2}} + \frac{63}{2816}x^{\frac{11}{2}}$ &c.
sive $x^{\frac{1}{2}}$ in $1 + \frac{1}{6}x + \frac{3}{40}x^2 + \frac{5}{112}x^3 + \frac{35}{1152}x^4 + \frac{63}{2816}x^5$, &c.
Non secus ponendo CB esse x, & radium CA esse 1, invenies Arcum LD esse $x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7$, &c.

Inventio Basis ex data Longitudine Curvae.

Si ex dato arcu αD Sinus AB desideratur; aequationis $z = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7$, &c. supra inventae, (posito nempe AB = x, αD = z, & Aα = 1,) radix extracta erit $x = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9$, &c.

Figure 3 Newton's arcsine and sine series in the *De analysi*

The potential pitfalls of historical research, however, did not discourage most of its practitioners. Moreover, descriptions of interesting earlier developments not only satisfied historical curiosity but sometimes spurred significant new mathematical discoveries. For instance, in the mid-eighteenth century Abraham Kästner's study of medieval and Renaissance arguments concerning the parallel postulate sparked renewed interest in foundations of geometry that ultimately inspired development of non-Euclidean geometries [7, p. 113; 9, pp. 154–156]. Increased historical knowledge and greater historiographic sophistication eventually produced a broadly based consensus about expectations for serious scholarship in history of mathematics: researchers emphasized the need to investigate original sources, even in non-European languages, and take into account the historical and cultural contexts of their composition.

One concept, two histories

The parallel evolution of mathematical knowledge in separate cultural traditions complicates the task of chronicling the origins of particular concepts or techniques. Developments familiar to us under the name of a classical or modern mathematician may have emerged earlier in a different culture, leading to rather anachronistic-sounding statements like "Babylonian scribes in the second millennium BCE used the Pythagorean theorem" or "the Fibonacci sequence appears in medieval Indian combinatorics". The principle that mathematical knowledge is fundamentally universal underneath its different linguistic and methodological guises has important consequences for the overall narrative of its history.

The rest of this section comprises a case study examining such consequences as they relate to current historiographic debates about how to define and identify the event(s) we know as 'the invention of calculus': specifically, the use of infinitesimal techniques to furnish a power series expression for the sine. Two different mathematicians in very different historical circumstances came up with what is indisputably the same result, which however has highly disputable implications for situating the origins of calculus within the history of mathematics.

Newton's sine power series

In early modern Europe, the development of 'new analysis' or infinitesimal calculus methods included the discovery of infinite series expressions for many transcendental functions, whose derivatives or integrals thus could likewise be expressed as infinite series. Among these results were the trigonometric power series published in 1711 in Isaac Newton's *De analysi per aequationes numero terminorum infinitas* ('Analysis by means of equations with an infinite number of terms', Figure 2) but worked out by Newton several decades earlier. Figure 3 shows the series expressions for the arcsine and sine expressions (titled 'To find the lengths of curves' and 'Finding the base [sine] from a given length of a curve' respectively) as Newton wrote them. Newton's diagrams are facsimiles from [18, pp. 15–17]; the simplified version in Figure 4 and the accompanying derivation are adapted from [8, pp. 5–19]. Following standard conventions of trigonometry before the eighteenth century, Newton interprets 'sine' as the length of a line segment rather than as the ratio of two lengths. In Newton's equations transcribed below, $\arcsin(x)$ represents the circular arc LD (with unit radius) corresponding to the sine CB in the first figure, while $x = \sin(z)$ is the sine AB of the arc αD in the second figure:

$$\arcsin(x) = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \cdots$$
$$\sin(z) = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9 - \cdots.$$

The determination of the arcsine series is more easily seen from the simpler diagram in Figure 4, where the sine of the arc z is the horizontal line segment x, which with the perpendicular segment y and the radius 1 forms a right triangle. An infinitesimally small 'moment' or increment of x is labeled dx (in a Leibnizian notation that the modern reader will probably find more familiar than Newton's), and the corresponding increment in the arc z is called dz, which can also stand for the rectilinear hypotenuse of the infinitesimal right triangle whose base is dx.

Since the tangent line containing the hypotenuse dz is perpendicular to the radius forming the larger triangle's hypotenuse, the two right triangles may be considered similar and consequently

$$\frac{dz}{1} = \frac{dx}{y} = \frac{dx}{\sqrt{1-x^2}}.$$

Newton can then rewrite the square-root term as $(1+(-x^2))^{-1/2}$ and apply his well-known algorithm for expanding a rational power of a binomial:

$$\begin{aligned} dz &= dx(1+(-x^2))^{-1/2} \\ &= \left(1+\frac{1}{2}x^2+\frac{3}{8}x^4+\frac{5}{16}x^6+\cdots\right) dx. \end{aligned}$$

The right side is easy to integrate term by term to give an infinite series for the desired arcsine z:

$$\begin{aligned} z &= \arcsin(x) \\ &= x+\frac{1}{6}x^3+\frac{3}{40}x^5+\frac{5}{112}x^7+\cdots. \end{aligned}$$

To find the corresponding infinite series for the sine, on the other hand, involves no infinitesimal geometry but rather an algebraic algorithm for solving what Newton calls 'affected' equations: i.e., polynomials involving more than one power of the unknown. His algorithm allows him to invert the previous series expression and express x as a power series in z rather than vice versa, producing the familiar 'Newton power series' for the sine [6, 18]:

$$\begin{aligned} x &= \sin(z) \\ &= z-\frac{1}{6}z^3+\frac{1}{120}z^5-\frac{1}{5040}z^7+\cdots. \end{aligned}$$

Mādhava's sine power series

Over two hundred and fifty years before Newton's work, a Hindu Brāhmaṇa astronomer named Mādhava in south-west India had begun investigating ways to represent trigonometric quantities using techniques known in traditional Sanskrit mathematics as *saṅkalita*: literally 'summation', meaning among other things the computation of successive sums, i.e., operations with series. This scholar left no surviving independent texts on his mathematical discoveries, but later generations of his followers carefully preserved and expounded a selection of Sanskrit verses containing trigonometric results attributed to him. The rest of this section treats the explanation of one such result, a rule for computing the sine of a given arc, recorded by an author named Śaṅkara who wrote near the middle of the sixteenth century. (The exposition in these two sections is based on verses 348–436 of Śaṅkara's commentary on *Tantrasaṅgraha* 2 [26, pp. 109–117]. A fuller description of the reconstruction of the rationale is given in [21] , and the base-text itself is translated in [23].See [27] for a discussion of a similar rationale in a related text from Mādhava's school. The bibliographies of all the above works cite numerous other sources relating to these remarkable mathematical developments.)

The algorithm as Śaṅkara explains it makes use of a trigonometric circle whose radius is larger than unity (a standard feature in pre-modern Indian trigonometry). So in the discussion that follows we will capitalize the name of this quantity as the 'Radius' R, and the correspondingly scaled trigonometric function values as, e.g., $\mathrm{Sin}(x) \equiv R\sin(x)$ for some given arc x. Śaṅkara first states the algorithm as follows:

"Having multiplied the arc and the results of each [multiplication] by the square of the arc, divide by the squares of the even [numbers] together with [their] roots, multiplied by the square of the Radius, in order. Having put down the arc and the results one below another, subtract going upwards. At the end is the Sine..."

This rather cryptic procedure should be interpreted as a recursive formula using squares of successive positive even integers and the square of R to produce coefficients for successive odd powers of x. The computation of the first three of these terms is as follows, shown in modern notation:

$$\begin{aligned} \frac{x^3}{R^2(2^2+2)} &= \frac{x^3}{R^2\cdot 6}, \\ \frac{x^5}{R^4(2^2+2)(4^2+4)} &= \frac{x^5}{R^4\cdot 120}, \\ \frac{x^7}{R^6(2^2+2)(4^2+4)(6^2+6)} &= \frac{x^7}{R^6\cdot 5040}. \end{aligned}$$

When we 'subtract going upwards', i.e., starting with the last of these calculated terms and ending with the given arc x itself, we obtain to following series expression:

$$\mathrm{Sin}(x) = x-\left(\frac{x^3}{R^2\cdot 6}-\left(\frac{x^5}{R^4\cdot 120} - \left(\frac{x^7}{R^6\cdot 5040}-\cdots\right)\right)\right).$$

Since the circular arc x is conventionally expressed in arcminutes of which there are 21600 in the circumference, and Mādhava's standard value of R is about $3438'$ or approximately equal to $21600/(2\pi)$, it is evident that dividing x by R essentially just converts x to radian measure. That is, this rule for the sine is exactly equivalent to the power series later found by Newton:

$$\sin(x) = x-\frac{x^3}{6}+\frac{x^5}{120}-\frac{x^7}{5040}+\cdots.$$

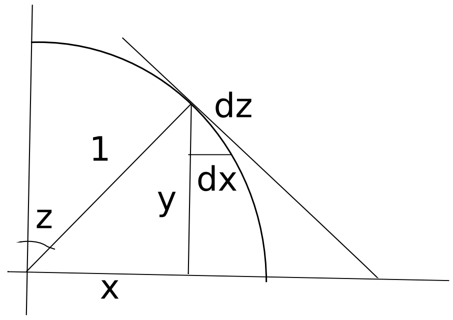


Figure 4 Line segments and their small 'moments' form 'similar' triangles

Recapitulating Mādhava's derivation

The rationales for this rule explained by Śaṅkara and other followers of the Mādhava school are based on the notion of dividing a quadrant into some integer number n of equal arcs, here denoted α, whose Chord is abbreviated to $\mathrm{Crd}(\alpha)$ or Crd. We will represent Śaṅkara's exclusively verbal exposition symbolically by denoting the Sine and Cosine of each successive multiple $k\alpha$ ($k = 1$ to n) of the standard arc α by $\mathrm{Sin}_k \equiv \mathrm{Sin}(k\alpha)$ and $\mathrm{Cos}_k \equiv \mathrm{Cos}(k\alpha)$, respectively. His derivation also involves Sines and Cosines of arcs extending to the midpoint of each of the equal arcs α, that is, arcs equal to $(k\alpha+\alpha/2)$ for $k = 0$ to $(n-1)$ and denoted by the subscript $k.5$:

$$\begin{aligned} \mathrm{Sin}_{k.5} &\equiv \mathrm{Sin}(k\alpha+\alpha/2), \\ \mathrm{Cos}_{k.5} &\equiv \mathrm{Cos}(k\alpha+\alpha/2). \end{aligned}$$

The differences between function values for two successive arcs are defined as follows:

$$\begin{aligned} \Delta\,\mathrm{Sin}_k &\equiv \mathrm{Sin}_k-\mathrm{Sin}_{k-1}, \\ \Delta\,\mathrm{Cos}_k &\equiv \mathrm{Cos}_{k-1}-\mathrm{Cos}_k, \\ \Delta\,\mathrm{Sin}_{k.5} &\equiv \mathrm{Sin}_{k.5}-\mathrm{Sin}_{k.5-1}, \\ \Delta\,\mathrm{Cos}_{k.5} &\equiv \mathrm{Cos}_{k.5-1}-\mathrm{Cos}_{k.5}. \end{aligned}$$

Figure 5 shows the quadrant divided into n equal arcs α and some of the line segments that represent differences between successive Sines and Cosines.

The similarity of the various right triangles composed of such line segments defines

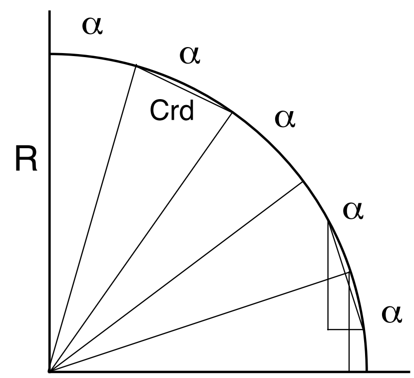


Figure 5 Mādhava's quadrant divided into equal arcs α

the following relationships between these quantities:

$$\begin{aligned}\Delta\,\mathrm{Sin}_{k.5} &= \mathrm{Cos}_k\cdot\frac{\mathrm{Crd}}{R},\\ \Delta\,\mathrm{Cos}_{k.5} &= \mathrm{Sin}_k\cdot\frac{\mathrm{Crd}}{R},\\ \Delta\,\mathrm{Sin}_{k+1} &= \mathrm{Cos}_{k.5}\cdot\frac{\mathrm{Crd}}{R},\\ \Delta\,\mathrm{Cos}_{k+1} &= \mathrm{Sin}_{k.5}\cdot\frac{\mathrm{Crd}}{R}.\end{aligned}$$

Note that up to now no arguments involving 'infinitesimal geometry' are employed: all these relations involving finite quantities are geometrically exact, with Cosines proportional to differences of Sines and vice versa.

Śaṅkara's next step is to reformulate key expressions in terms of sums of Sine- and Cosine-differences. For example, the kth Sine itself is trivially just the sum of k successive Sine-differences:

$$\mathrm{Sin}_k = \Delta\,\mathrm{Sin}_1 + \Delta\,\mathrm{Sin}_2 + \cdots + \Delta\,\mathrm{Sin}_k\,.$$

From the previous similar-triangle equations, we derive the following expression for the differences of the successive Sine-differences themselves, or 'Sine second-differences':

$$\begin{aligned}&\Delta\,\mathrm{Sin}_k - \Delta\,\mathrm{Sin}_{k+1}\\ &\quad= \mathrm{Cos}_{k.5-1}\cdot\frac{\mathrm{Crd}}{R} - \mathrm{Cos}_{k.5}\cdot\frac{\mathrm{Crd}}{R}\\ &\quad= \Delta\,\mathrm{Cos}_{k.5}\cdot\frac{\mathrm{Crd}}{R}\\ &\quad= \mathrm{Sin}_k\cdot\frac{\mathrm{Crd}^2}{R^2}.\end{aligned}$$

That is, the Sine second-difference is proportional to the corresponding Sine. This allows the desired Sine-difference to be rewritten tautologically as

$$\Delta\,\mathrm{Sin}_{k+1} = \Delta\,\mathrm{Sin}_k - (\Delta\,\mathrm{Sin}_k - \Delta\,\mathrm{Sin}_{k+1}).$$

Each of the previous Sine-differences can be rewritten in the same way, until the desired Sine-difference is expressed as the first Sine-difference minus a sum of k second-differences or equivalent quantities, as follows:

$$\begin{aligned}\Delta\,\mathrm{Sin}_{k+1} &= \Delta\,\mathrm{Sin}_{k-1} - (\Delta\,\mathrm{Sin}_{k-1} - \Delta\,\mathrm{Sin}_k)\\ &\qquad - (\Delta\,\mathrm{Sin}_k - \Delta\,\mathrm{Sin}_{k+1})\\ &= \Delta\,\mathrm{Sin}_1 - \cdots - (\Delta\,\mathrm{Sin}_k - \Delta\,\mathrm{Sin}_{k+1})\\ &= \Delta\,\mathrm{Sin}_1 - \sum_{j=1}^{k}(\Delta\,\mathrm{Sin}_j - \Delta\,\mathrm{Sin}_{j+1})\\ &= \Delta\,\mathrm{Sin}_1 - \sum_{j=1}^{k}\mathrm{Sin}_j\cdot\frac{\mathrm{Crd}^2}{R^2}\\ &= \Delta\,\mathrm{Sin}_1 - \sum_{j=1}^{k}\Delta\,\mathrm{Cos}_{j.5}\cdot\frac{\mathrm{Crd}}{R}.\end{aligned}$$

Given the above expression for each individual Sine-difference, one can sum up the sums to get an expression for the desired Sine:

$$\begin{aligned}\mathrm{Sin}_k &= \Delta\,\mathrm{Sin}_1 + \Delta\,\mathrm{Sin}_2 + \cdots + \Delta\,\mathrm{Sin}_k\\ &= \Delta\,\mathrm{Sin}_1 + \Delta\,\mathrm{Sin}_1 - \sum_{j=1}^{2}\mathrm{Sin}_j\cdot\frac{\mathrm{Crd}^2}{R^2}\\ &\qquad + \cdots + \Delta\,\mathrm{Sin}_1 - \sum_{j=1}^{k-1}\mathrm{Sin}_j\cdot\frac{\mathrm{Crd}^2}{R^2}\\ &= k\Delta\,\mathrm{Sin}_1 - \sum_{j=1}^{k-1}\sum_{g=1}^{j}\mathrm{Sin}_g\cdot\frac{\mathrm{Crd}^2}{R^2}\\ &= k\Delta\,\mathrm{Sin}_1 - \sum_{j=1}^{k-1}\sum_{g=1}^{j}\Delta\,\mathrm{Cos}_{g.5}\cdot\frac{\mathrm{Crd}}{R}.\end{aligned}$$

That is, each Sine is represented by its index number k times the first Sine-difference, minus a double sum of either successive Sines or successive Cosine-differences.

At this point, the demonstration departs from geometric exactness by making some simplifying assumptions about the size of the quantities involved. These assumptions depend on letting the constant arc-increment α be very small, on the order of one arcminute. This allows us to take α approximately equal to its own Chord, and also to the first Sine-difference.

Likewise, we eliminate half-subscripts by ignoring the distinction between the endpoints and midpoints of the successive arc-increments. Finally, we assume that the given arc whose Sine Sin_k is sought is approximately equal to k arcminutes, and ignore the difference of one arcminute between k and $k-1$. These assumptions allow us to rewrite the above expressions for Sin_k more concisely, as follows:

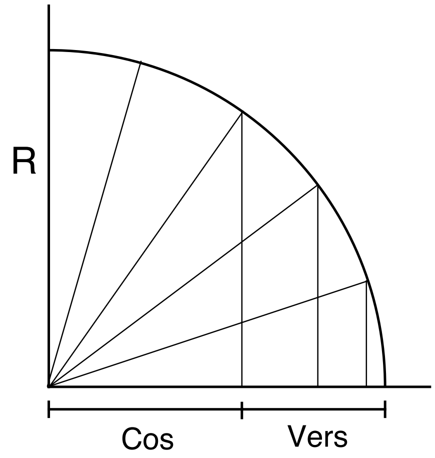


Figure 6 The Versine as a sum of Cosine-differences

$$\begin{aligned}\mathrm{Sin}_k &\approx k - \sum_{j=1}^{k}\sum_{g=1}^{j}\mathrm{Sin}_g\cdot\frac{1}{R^2}\\ &\approx k - \sum_{j=1}^{k}\sum_{g=1}^{j}\Delta\,\mathrm{Cos}_g\cdot\frac{1}{R}.\end{aligned}$$

To understand Śaṅkara's next step, we have to employ a trigonometric identity that would have been obvious to him but has fallen into disuse in modern trigonometry. This is the relationship between the Cosine and the so-called 'versed sine' or Versine, defined as the difference between the Radius and the Cosine (see Figure 6). Consequently, the cumulative differences between all successive Cosines up to the kth Cosine add up to the kth Versine. This identity lets us transform the above double sum of Cosine-differences into a single sum of Versines:

$$\begin{aligned}\mathrm{Sin}_k &\approx k - \sum_{j=1}^{k}\sum_{g=1}^{j}\mathrm{Sin}_g\cdot\frac{1}{R^2}\\ &\approx k - \sum_{j=1}^{k}\sum_{g=1}^{j}\Delta\,\mathrm{Cos}_g\cdot\frac{1}{R}\\ &\approx k - \sum_{j=1}^{k}\mathrm{Vers}_j\cdot\frac{1}{R}.\end{aligned}$$

Pausing a moment to assess the status of this rationale — which, we recall, is supposed to justify the above-mentioned 'Mādhava–Newton power series' expression for the sine — we may feel that not much progress has been made. The unknown Sine is now expressed in terms of a sum of k unknown Versines, and we seem no closer to knowing how to compute it. Overcoming this difficulty will require a brief detour to infer a pattern in expressing partial sums of powers of successive integers.

Basic number theory formulas for, e.g., the sum of successive positive integers from 1 to some arbitrary k and the sum of their squares up to k^2 had been known in India for many centuries. Śaṅkara now assumes such a k sufficiently large to reduce all such formulas to the same general approximate form, as follows:

$$\sum_{j=1}^{k} j = \frac{k(k+1)}{2} \approx \frac{k^2}{2},$$
$$\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6} \approx \frac{2k^3}{6} = \frac{k^3}{3},$$
$$\vdots$$
$$\sum_{j=1}^{k} j^n \approx \frac{k^{n+1}}{n+1}.$$

Then if each such sum of powers is itself summed, the resulting *saṅkalita* or sum-of-sums expressions can be represented in the form

$$\sum_{j=1}^{k} \frac{j^2}{2} \approx \frac{k^3}{6}, \quad \sum_{j=1}^{k} \frac{j^3}{6} \approx \frac{k^4}{24}, \quad \sum_{j=1}^{k} \frac{j^4}{24} \approx \frac{k^5}{120},$$

and so on.

These closed-form approximations for partial sums can be substituted for sums of trigonometric functions by making further simplifying assumptions. In the equation for the desired Sine, Śaṅkara considers each successive Sine Sin_g approximately equal to its corresponding arc, namely g arcminutes:

$$\begin{aligned} \mathrm{Sin}_k &\approx k - \sum_{j=1}^{k} \mathrm{Vers}_j \cdot \frac{1}{R} \\ &\approx k - \sum_{j=1}^{k} \sum_{g=1}^{j} \mathrm{Sin}_g \cdot \frac{1}{R^2} \\ &\approx k - \sum_{j=1}^{k} \sum_{g=1}^{j} g \cdot \frac{1}{R^2}. \end{aligned}$$

Now by using the above general form for sums of sums of powers of integers, we can convert this equation to

$$\begin{aligned} \mathrm{Sin}_k &\approx k - \sum_{j=1}^{k} \sum_{g=1}^{j} g \cdot \frac{1}{R^2} \\ &\approx k - \sum_{j=1}^{k} \frac{j^2}{2} \cdot \frac{1}{R^2} \\ &\approx k - \frac{k^3}{6R^2}. \end{aligned}$$

This is the key step in arriving at the desired power series. Since the Sine and Versine are each defined in terms of a sum of the other, these *saṅkalita* terms can be recursively substituted for sums of trigonometric function values indefinitely. For instance, the previous approximation for the Sine allows us to rewrite the Versine as

$$\begin{aligned} \mathrm{Vers}_j &\approx \sum_{j=1}^{k} \left(j - \frac{j^3}{6R^2} \right) \cdot \frac{1}{R} \\ &\approx \frac{k^2}{2R} - \frac{k^4}{24R^3}. \end{aligned}$$

And that expression for the Versine in turn may be plugged back in to the formula for the Sine:

$$\begin{aligned} \mathrm{Sin}_k &\approx k - \sum_{j=1}^{k} \left(\frac{j^2}{2R} - \frac{j^4}{24R^3} \right) \cdot \frac{1}{R} \\ &\approx k - \frac{k^3}{6R^2} + \frac{k^5}{120R^4}. \end{aligned}$$

The same pattern of alternating substitution between Sine and Versine terms can be extended indefinitely. And the above computation of the first few terms for the Sine indicates how this pattern ultimately produces the infinite series expression that we originally undertook to justify.

Implications for the history of calculus

It is clear from comparing even these brief and over-simplified sketches of the two discoveries of the sine power series that they are very different in context and style, although there is considerable overlap in the concepts they use. Newton in the mid-seventeenth century was interested in finding the most general methods possible to perform quadratures of diverse curves using simple polynomial integration techniques. The power series for the sine (along with the one for the arcsine) was essentially a by-product of this quest for general solutions in the *De analysi*.

Mādhava around the turn of the fifteenth century, as noted above, did not pass down to the present day any text describing in detail the context of his work on infinite series for trigonometric functions, though it seems reasonable to assume that his successors followed his lead in the way they presented it. What is evident is that Mādhava saw this work as dealing specifically with the computation of Sines and arcs. The proportion linking Sines and Cosine-differences was not a particular application of a general concept like fluxions or derivatives to a function that happened to be trigonometric. Likewise, the relationship between positive integer powers of successive integers and their sums was not an instance of a general integration power law or a tool for quadrature of arbitrary curves. Still less did these ideas form part of any investigation of series composed from higher derivatives of arbitrary functions, as in the modern Taylor series. This is further confirmed by later derivations, apparently due to Śaṅkara himself, that employ related techniques to infer a different sine power series that is in some ways inconsistent with the Mādhava–Newton one [22]. Rather, these algorithms in the work of Mādhava's school remained fundamentally embedded in the trigonometry of Sines and their associated Chords, Cosines and Versines.

However, the similarities in the results found by Mādhava and Newton, and also in the basic structure of some of the tools they used (geometric approximations based on infinitesimally small increments, recursive substitutions for terms of series, and so forth), are significant enough to warrant questioning some of our default classifications of historical developments in mathematics. Mathematicians and historians of mathematics generally take it for granted that 'calculus' refers to the infinitesimal analysis of derivatives and integrals that took shape in seventeenth-century Europe, most significantly in the work of Newton and Leibniz. If that is what 'calculus' means, then of course Mādhava cannot be assigned priority in discovering it. But if 'calculus' is used to connote a broader constellation of concepts involved in those discoveries, then it seems undeniable that Mādhava and his followers were in some sense doing calculus too. A strong version of the latter perspective appears in many discussions of the Mādhava school, e.g., in a foreword to a recent translation of one of its seminal treatises asserting 'The origin of calculus also can be traced to this school' [23, p. vii]. (Contrast a slightly earlier version of the former perspective that opens with the equally blunt assertion 'Calculus was not invented in India' [1, p. 131].) A more extreme approach argues for the possibility that the discoveries of the Mādhava school actually exerted crucial influence on early modern European calculus via transmission during the colonial period; e.g., [14, pp. 178–204].

Such unexpected overlaps between historically distinct mathematical traditions are valuable in correcting the automatic tendency of historians to essentialize certain developments or methodologies primarily because they happen to be familiar. Mādhava will probably not replace Newton and Leibniz in most future histories of mathematics as the iconic 'inventor of calculus'; but the fact that such a replacement might even be considered possible should provoke some salutary questioning about what exactly we mean when we speak of 'inventing calculus'.

The history of mathematics now

The discipline of history of mathematics at present is expanding in terms of both its historical content and its professional role.

The acknowledged subject matter of history of mathematics now extends from the earliest identifiable records of human quantitative thought up to the achievements of mathematicians still living and working today. In particular, it is generally considered to embrace not only literate mathematical traditions from every place and time in the world's history but also many quasi-mathematical cultural features such as ornamentation patterns and game strategies, often classified under the name 'ethnomathematics'. More and more archival sources, linguistic traditions and scientific instruments are being examined for the evidence they may yield about contemporary mathematical developments. Texts in related sciences such as astronomy and physics (and related pseudo-sciences such as astrology) are read side by side with 'pure' mathematical ones in the effort to understand the mathematics of earlier periods as contemporary teachers and students saw it.

Professionally, growing numbers of mathematicians read and publish research in history of mathematics, and growing numbers of secondary schools, colleges and universities include it in their curricula. Readers, teachers and researchers are increasingly attentive to the interdisciplinary requirements of the field. Its practice now has to incorporate research skills in history and languages as well as analyzing the technical content of the mathematics and related subjects it investigates.

Focal points in current history of mathematics

Although research in history of mathematics nowadays involves an immense variety of subjects, sources and methodologies, there are certain common themes that unite many of them in general trends. Perhaps the most important of these is the issue of what is sometimes called 'external history', or contextualizing technical knowledge within a particular historical and cultural setting. Modern mathematical notations and classifications, although they are useful tools for interpreting earlier writings, cannot be automatically accepted as reliable equivalents for them; nor can technical treatises really be understood in linguistic or cultural isolation from their counterparts in other fields. Two scholars of the oldest known literate mathematical traditions have well described the recent shift in approaches to reading ancient mathematical texts: (see also the discussions of ancient source interpretation in [12]):

"The study of Egyptian mathematics is as fascinating as it can be frustrating. The preserved sources are enough to give us glimpses of a mathematical system that is both similar to some of our school mathematics, and yet in some respects completely different. It is partly this similarity that caused early scholars to interpret Egyptian mathematical texts as a lower level of Western mathematics and, subsequently, to 'translate' or rather transform the ancient text into a modern equivalent. This approach has now been widely recognized as unhistorical and mostly an obstacle to deeper insights. Current research attempts to follow a path that is sounder historically and methodologically. Furthermore, writers of new works can rely on progress that has been made in Egyptology (helping us understand the language and context of our texts) as well as in the history of mathematics." [13, p. 7]

"It is tempting to think that, because it all happened such a long time ago, there is little new to say about mathematical developments in ancient Mesopotamia (southern Iraq and neighbouring areas). The standard histories of mathematics all tell much the same story... [F]amously the young Otto Neugebauer began his program of decipherment and publication in the late 1920s... Neugebauer's interpretative paradigm remained paramount: analysis in mathematical terms only, highlighting features such as the use of the Pythagorean theorem that could be taken as an index of Babylonian progression toward modernity. Questions of authorship, context, and function were systematically overlooked; textuality and materiality played no part in the academic discourse of the mid-twentieth century... Mathematics is not created out of nothing — it is written by individuals operating within the social and intellectual norms and conventions of the societies in which they dwell. Thus coming to grips with another culture's mathematics is not simply a matter of translating one notation into another. Instead we need to explore the personal, intellectual and social circumstances under which it was written." [25, pp. 58–62]

A related trend is the growing emphasis among scholars on the importance of original source materials in historical research and the need for better access to them. Archives of publications by individual mathematicians and scientific academies are appearing at every level from nationally funded institutions to editions of collected works to small textbase projects maintained by individuals. Improved digital availability is rapidly accelerating the dissemination of historical sources, assisted by a steady stream of new translations produced by both researchers and students.

For example, the Euler Archive (www.eulerarchive.org) is a pioneering online archive collecting not only facsimile publications but new translations and bibliographies of Euler's work, as well as other information about him. The Galileo Portal (portalegalileo.museogalileo.it) is an instance of a similarly structured but more ambitious repository based at the Florence Institute and Museum of the History of Science. A smaller-scale version of such an archive devoted to the mathematician Johann Lambert is being developed at www.kuttaka.org/JHL/Main.html. The various mathematical bibliographies and textbases available are too numerous to list; a sketch of relevant electronic resources, now nearly ten years old but still informative about several active projects, is [7, pp. 337–338].

These developments accompany, and in many cases are inspired by, the continued symbiosis between mathematics history and mathematics pedagogy: instructors and students alike are seeking source materials, research topics, presentation opportunities, and above all more sophisticated and detailed answers to historical questions than are found in long-established standard textbooks, whose surveys are often decades out of date in certain areas. The difficulty of finding accessible and accurate secondary sources on many historical topics has strengthened support for 'do-it-yourself' research on historical sources for both teachers and students. For example, a Special Interest Group of the Mathematical Association of America currently directs an annual student writing contest in the history of mathematics (www.homsigmaa.org); there are also various workshops and technical guides aimed at novices to the study of original sources, notably [34].

Finally, at the specialist research level, common ground is beginning to be negotiated on what might be called the historiography of 'world mathematics'. As discussed in the previous sections, the theoretical ideal of mathematics as universal knowledge, colliding with the varied ways that different cultures have defined and pursued it, can lead to confusion or controversy about how to identify and attribute discoveries (and re-discoveries) of important results. Researchers in diverse subfields are attempting to establish new demarcation lines (or in some cases to eradicate old ones) defining different linguistic/cultural traditions, time periods, and more abstract concepts like 'proof' or 'exact sciences'. Thoughtful discussion is steadily supplanting the exaggerated inferences and mutual suspicions of cultural chauvinism that had sometimes hampered communication on such questions. Two such discussions treating the notion of proof in Indian mathematics are [17] and [27, Vol. I, pp. 267–310]. A similar exploration of proof modalities in Chinese mathematics is [5]; a collection of cross-cultural assessments of these issues will appear shortly [4].

Probable directions for the future

Astrology is no longer considered a subfield of mathematics, and consequently we do not expect mathematicians to show any more skill than the average when it comes to predicting the future. Yet it seems appropriate to close a wide-ranging survey of this sort by offering at least some tentative indications of what is likely to happen next in the history of mathematics, in addition to the continuance of the current general trends mentioned above. One of the most promising new avenues of research appears to be the investigation of mathematical corpora in cultural or linguistic minority populations, which tend to be overshadowed by the dominance of Latin, French, German, Italian and English among the Western languages, and Arabic, Chinese and (to a lesser extent) Sanskrit within Asia. Under-studied but potentially very rich fields include South Asian vernacular mathematics in the second millennium, Dutch mathematical sciences in the early modern period, and eastern European mathematics in the twentieth century.

History of mathematics will probably take a more prominent part in popular perceptions of mathematics, both in and outside classrooms. Several dramatic works in recent years have highlighted the significance of mathematics in the life of the mind, as have new outreach efforts to encourage public interest in mathematics. The creation of the New York Museum of Mathematics (momath.org, opening in 2012) is a striking example of this growing attention to public awareness of mathematics. The role of history in all such programs of math popularization is likely to be crucial.

Without doubt, though, its greatest impact will continue to be felt in mathematics teaching. In fact, the chief danger now for mathematics and history of mathematics alike is not that the latter will be neglected by the former, but that it may be overloaded with pedagogical expectations to achieve what mathematics itself has not: namely, providing an easy solution to the challenges of teaching mathematical thinking to students for whom technology has largely replaced basic mathematical competence. Nonetheless, although the integration of mathematics with humanities and social sciences that the history of mathematics offers cannot eliminate all difficulty from learning or teaching mathematics, it remains an invaluable reminder of the importance of overcoming that difficulty.

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