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Education

Linear algebra with a didactical focus

How might you construct an introductory linear algebra course for first year mathematics students? What decisions would you have to make and what issues would you have to address? Barbara Jaworski, Stephanie Treffert-Thomas and Thomas Bartsch, as a small research team, set out to address these questions and others relating to a first year, first semester module in linear algebra. The authors are all members of the School of Science at Loughborough University, they all teach mathematics and do research into mathematics or mathematics education. Thomas Bartsch is a mathematician working in the Department of Mathematical Sciences; Barbara Jaworski and Stephanie Treffert-Thomas are mathematics educators working in the Mathematics Education Centre.

The Mathematics Education Centre (MEC) was opened in 2002 to provide university wide support for students engaging with mathematics in any disciplinary area of the university. It includes two drop-in Mathematics/Statistics Learning Support Centres which are staffed by a mathematician and/or statistician for six or seven hours each day. Members of the MEC do research into mathematics learning and teaching, primarily at university level. They contribute to mainstream teaching of mathematics and provide expertise in teaching mathematics to engineering students.

Background to the study

An aim in studying the teaching of linear algebra was to try to start to characterise mathematics teaching within the university and to gain access to the perspectives of mathematicians on their teaching of mathematics. A seminar series (entitled 'How we Teach') had been started to share aspects of mathematics teaching and initiate a mathematics teaching discourse through which we could learn from each other and develop our teaching. Seminars in the series were video recorded and a selection of them analysed in order to characterise this discourse [3]. Seminars form a part of the New Lecturer's Course for new mathematics lecturers at Loughborough.

The research study was agreed between Barbara Jaworski and Thomas Bartsch before the start of the academic year 2008/09. Stephanie Treffert-Thomas joined the team as a PhD student with this research the focus of her PhD. Thomas Bartsch was in his second year of teaching this module. Barbara Jaworski had considerable experience of doing research into mathematics teaching at a variety of levels. Together the team formed a small community of inquiry. We had a common purpose in exploring the teaching of mathematics, trying to understand better the teaching process, recognising the issues which arise for teacher and students, and promoting development of teaching. We had differing roles with Thomas Bartsch as lecturer, having responsibility for design of the module and module materials, teaching and monitoring students, and Barbara Jaworski and Stephanie Treffert-Thomas as re-

A tutor helping a student

Two students working together

searchers, having responsibility for conducting research of a largely qualitative nature.

Our research methodology was ethnographic in style, that is producing qualitative data through conversations and interviews. It was important for the two researchers to gain in-depth access to the thinking and actions of the lecturer in order to develop well-grounded understandings of the lecturer's teaching and design of teaching. Thus the two researchers talked extensively with the lecturer before and after lectures, observed all lectures and tutorials, and collected relevant documents. In addition Stephanie Treffert-Thomas sought students' views with two questionnaires handed out in lecture time and by conducting focus group interviews with a small number of students when the first semester teaching had ceased. Research meetings of the team and all teaching by the lecturer were audio-recorded. Analysis of this data was qualitative, involving repeated listening, transcribing, coding and categorising. Atlas-ti software was used extensively to support analysis.

The first semester linear algebra module

Linear algebra is a mainstream topic for first year mathematics students. It is taught in a two-semester module with 72 hours of teaching and associated assignments and examination. Thomas Bartsch is the lecturer for the first semester (S1); there is a different lecturer in the second semester (S2). The two lecturers collaborate on the year-long design of the module and prepare a joint examination at the end of the year. The first semester offers an introduction to linear algebra and the second semester a more abstract treatment. In this study we focus on the first semester which consists of an introduction to linear algebra that tries to avoid the more formal aspects of the material. The second semester involves a repetition of the same material, but from a formal perspective. One purpose of such organisation is to recognise that students coming to university from school are not well prepared for mathematical formalism (see, for example, [4]) and need some preparation for dealing with abstraction. The module is taught through two lectures and one tutorial each week (the standard allocation of time).

The module that we observed was taught to a cohort of 240 students of which approximately 180 (based on informal, periodic head counts) attended lectures regularly. The lecturer distributed weekly problem sheets on which students were asked to work in their own time. In addition, each student is a member of a Small Group Tutorial (SGT) in which seven or eight students meet once a week with a tutor who is a mathematics lecturer (not a graduate student). (In the UK, the academic hierarchy is Lecturer, Senior Lecturer, Reader, Professor. Most academics are at the levels of Lecturer or Senior Lecturer. The term 'lecturer' is used both as an academic title and as the role of the academic teaching a particular module.) In SGTs some of the tutorial problems could be discussed, at the discretion of the tutors and their students. For all problems the lecturer made detailed solutions available after two weeks. SGT tutors are also personal tutors for students in their group. Through the SGTs they have access to student progress and student experiences of learning and teaching.

The lecturer's design of the module included choosing, sequencing and writing the mathematical content, including the examples used in lectures and the examples/exercises used in the weekly tutorial, designing a weekly problem sheet, and preparing assessment tasks which included on-line tests and written coursework. In the first semester, the lecturer prepared notes-with-gaps which were placed on LEARN (a virtual learning environment) for students to access in advance of a lecture.

The lecturer's notes were structured to guide the course and were used for teaching; that is they were presented to students by the lecturer in each lecture. Students were asked to bring printed copies of the notes to the lecture. Tutorials differed from lectures by focusing only on examples with no progression of the material of the notes. The lecturer used a data projector to project the course notes, including the outline of examples, onto a big screen, and an overhead projector to work out the solutions to examples, which were missing from the printed notes. He would move physically between the two. Often he stood centrally in the lecture theatre to talk to the students offering his own comments about the mathematics and about ways in which students should approach the mathematics.

A tutor helping a student

One purpose of the gaps in the lecture notes was to encourage students to attend lectures and complete the notes *in the lecture*. This involved completing the *solutions* of key examples that were presented. Often, before presenting a solution, the lecturer gave students some minutes to work on the solution by themselves or with their neighbours, walking around the lecture theatre and talking with some students.

The design of the module gave students the option to engage with the content of the module in a variety of ways. They could download the lecture notes from LEARN. They could attend lectures and tutorials, fill in the gaps in the notes and make their own supplementary notes, attend their own SGT each week, and get access to the lecturer either face to face or by email. They could work on problem sheets and complete assignments marked by their SGT tutor. The SGT provides opportunity for discussion with fellow students, and the lecturer encouraged such discussions also outside of the formal teaching sessions. Students could also attend a support centre and get advice from a lecturer who was not otherwise involved in teaching the module.

The content of the first semester was presented in the course notes in four chapters as follows:

1. Linear Equation Systems
2. Matrices
3. Subspaces of $\mathbb{R}^n$
4. Eigenvalues and Eigenvectors

In Chapter 1 the focus was linear equation systems. The lecturer distinguished systems

A tutor helping a student

Students attending a lecture

of linear equations that have one, many or no solutions. He introduced the method of Gaussian elimination to determine the solution set of an arbitrary linear equation system. This method uses elementary row operations on a linear equation system, or its coefficient matrix, in order to produce an equivalent, but simpler system. Gaussian elimination is sometimes also referred to as the method of row-reduction of matrices. Chapter 2 consisted of an introduction to matrices as representing linear equation systems. The content in Chapter 2 included calculating with matrices (namely the addition, subtraction and multiplication of matrices), finding the inverse and the transpose of a given matrix, and the related rules of matrix algebra.

In the lecturer's own words Chapters 1 and 2 contained the more computational aspects of the module. These two chapters provided students with the necessary computational skills to advance to Chapters 3 and 4, which focused more strongly on concepts.

Chapter 3 dealt with the most important concepts in linear algebra, which are vector spaces, subspaces, span and spanning sets, range, linear independence, basis and dimension, and the rank-nullity theorem. These concepts were all introduced in the setting of $\mathbb{R}^n$. The lecturer presented examples and deduced general observations from the examples. Theorems were often presented as 'Observations' and in general, no abstract proofs were given throughout the first semester. (There were one or two exceptions.) This was a deliberate strategy employed by the lecturer and one that we discuss further below.

The focus in Chapter 4 was eigenvalues and eigenvectors. Chapter 4 included the definition of an eigenvector/value, an introduction to the theory of determinants, the use of the characteristic polynomial in calculating eigenvalues (and hence for finding eigenvectors), and a detailed account of the process of diagonalisation.

The nature of research meetings

Research meetings focused on the lecturer's design, planning and intentions for teaching. The meetings provided an opportunity for the lecturer to talk about his design of the module, his current teaching and perceptions of students' learning and issues arising thereof. The two observers asked questions and offered observations or perceptions. Meetings following a lecture or tutorial focused on what had taken place, and involved the lecturer's reflections interspersed with questions from the observers. Often our discussions in meetings focused on students' responses to the material and the lecturer's perception of students' understanding in relation to the material of the lecture. The nature of these discussions included the lecturer talking about his own conceptions of the material of the lecture, of his didactical thinking with regard to this material, of his perceptions of students' activity and of his decision-making in constructing notes, examples and assessment tasks. The example below, of the lecturer's talk, shows 'expository mode' (talking about his own conceptions of the material) in normal text and 'didactic mode' (talking about his construction of the teaching of the material) in italic text.

"Thursday is about defining the characteristic polynomial, understanding that its zeroes are the eigenvalues, and I'll show an

Example. 3.14. Consider an unknown 2×3 matrix A. We know that A satisfies $A\mathbf{x_1} = \mathbf{b_1}$ and $A\mathbf{x_2} = \mathbf{b_2}$, where

$$\mathbf{b_1} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad \mathbf{b_2} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}, \quad \mathbf{x_1} = \begin{pmatrix} 1 \\ 3 \\ -7 \end{pmatrix}, \quad \mathbf{x_2} = \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix}.$$

a. Is $\mathbf{b_1}$ in the range of A? Is $\mathbf{b_2}$ in the range of A?

b. Is $\mathbf{b_1} + \mathbf{b_2} = \begin{pmatrix} 1 \\ 8 \end{pmatrix}$ in the range of A?

c. Take the number $\lambda = 3$. Is $\lambda\mathbf{b_1} = \begin{pmatrix} 6 \\ 9 \end{pmatrix}$ in the range of A?

d. Is the zero vector $\mathbf{o}$ in the range of A?

Figure 1 An example offered to students in the module

example of an eigenvalue that has algebraic and geometric multiplicity 2. Algebraic multiplicity, meaning this is the power with which the factor lambda minus eigenvalue appears in the characteristic polynomial, and geometric multiplicity is the number of linearly independent eigenvectors. *And these are the important concepts for determining if a matrix is diagonalisable because, for that, we need sufficiently many linearly independent eigenvectors.* Now if an eigenvalue has algebraic multiplicity larger than 1, that means there are correspondingly fewer eigenvalues. So, in principle, we can fail to find as many eigenvectors as we need in that case. On the other hand, if an eigenvector has algebraic multiplicity 3, the geometric multiplicity can be anywhere between 1 and 3. If it's 3, we are fine, if it's less than 3, we're missing out at least one linearly independent eigenvector. And in such a case the matrix would not be diagonalisable. *And that's the big observation that we need to get at next week, that a matrix is diagonalisable if and only if all the geometric multiplicities are equal to the algebraic multiplicities."*

The distinction between expository mode and didactic mode is not clear cut. The sentence in italics in the middle of the quotation might also be characterised as expository mode. However, it seems here that the lecturer is *meta*-commenting on the material: i.e. expressing his value judgement regarding important concepts that need to be appreciated, rather than just articulating mathematical relationships. This seems to relate to didactic judgements in terms of what needs to be emphasised for students. We observe that such statements in meetings correspond to what we have called meta-comments, or meta-mathematical comments in lectures. Such comments address what students need to attend to, either in terms of their work on the mathematical content (meta-comments – A) or of their understanding of the mathematical content (meta-mathematical-comments – B). Examples A and B follow.

A: "First of all, ... if I give you an equation system, this gives you a recipe to decide if that equation system is consistent or inconsistent. You transform it to echelon form and you check if there is such a special row that makes the system inconsistent."

B: "But it's important that you be able to understand the language that we're using and to use it properly. So please, pay attention to the new terms and the new ideas that we're going to introduce over this chapter."

We are emphasising this difference in modes of talk about the material of the module to contrast thinking about teaching (the didactic mode) with thinking about mathematics (expository mode). In meta-comment A, the lecturer draws students' attention to the nature of the mathematics and how they work with it. In meta-mathematical comment B, he draws their attention to the processes of working with the mathematics and strategies that can lead to understanding. Both of these are 'didactical' approaches on the part of the lecturer. In studying the *teaching* of linear algebra, we are interested fundamentally in the didactic nature of the lecturer's presentation of the mathematics.

The lecturer's approach to teaching

From analysing the audio-recordings of the meetings between the lecturer and the two researchers, we gained insight into the lecturer's motivations, intentions and strategies for teaching. Based on his experience of teaching undergraduate mathematics for one year prior to this research, the lecturer devised an examples-based approach to the teaching of linear algebra for this module. In a research meeting, the lecturer said:

"Yes. .... Generally speaking, I decided that I would focus on doing the development of the argument on examples, and then trying to abstract a general fact from the example, as I have done in most cases so far. And so then, what I am doing is go through the example, and then highlight the important facts on the example, and then condense them into a general observation. And I have several times mentioned to students that this is what we're doing, and that it's a good idea to see an example not as an isolated example but rather as a representative of a big class. "

In taking this approach the lecturer 'avoided' the introduction of theorems although many of the 'observations' that he made were in fact equivalent to theorems. Few of the observations were proved in a formal sense.

We termed his approach EAG, where EAG stood for 'example–argument–generalisation'. The lecturer's approach could thus be summarised as:

- we introduce an **E**xample,
- we make an **A**rgument on the example, and then
- we **G**eneralise to an observation, another example or set of examples.

The term 'observation' above agreed with the use of this term in the lecture notes, where the lecturer used the term 'observation' rather than 'theorem'.

This approach could be described as 'bottom-up'. The lecturer demonstrated a mathematical phenomenon on a 'typical' example that served as a representative for a class of similar cases. He explained the example in a manner that was intended to highlight the general features rather than the specific details of the particular example. Where necessary, he introduced definitions to provide relevant terminology. General statements could then be abstracted from the arguments that were applied in the example. Because these statements arose from the study of an example they were called 'observations' rather than 'theorems', as they would be in more formal presentations of linear algebra.

The course covered all the standard results of introductory linear algebra. Because most of them were presented as observations that were justified by reasoning about an (typical) example, the first semester included hardly any formal proofs. The proofs were provided in the second semester, in which the results were revisited in the abstract context of vector space theory. By proceeding in this manner, the lecturer hoped to offer his students a gentle introduction to mathematical reasoning about objects and their properties that is required at university level.

An example-based approach as outlined above can be viewed in contrast to the more traditional ('top-down') deductive style of teaching mathematics at university. The latter is often referred to as DTP (definition–theorem–proof) or DLPTPC (definition–lemma–proof–theorem–proof–corollary) style (see, for example, [1, 5]). In a traditional approach (DTP), the statement "The range of a matrix is a subspace", for example, is introduced as a theorem. The theorem is then proved by checking that the three properties of a subspace (the set is closed under addition and scalar multiplication and contains the zero vector) are satisfied.

In our study, however, using the EAG approach, the lecturer set up an (concrete) example and asked a series of questions as shown in Figure 1. Earlier in the course, the lecturer had introduced the null space of a matrix A, i.e., the solution set of the homogeneous equation system $Ax = 0$. He had shown that the null space has similar properties to the set of all n-component vectors: It is closed under addition and scalar multiplication and contains the zero vector. This observation had motivated the definition of a subspace. The four questions (a) to (d) in the present example were designed to lead the student to recognise the correspondence between the answers to the questions and the definition of a subspace. As a result the students were to arrive at, and recognise that the range of a matrix is a subspace. This was then summarised in what the lecturer called 'Observation 3.15'. This 'observation' is the theorem "The range of a matrix is a subspace". The lecturer chose the terminology of 'Observation' (rather than 'Theorem') because he did not give a formal proof at this point in the course.

This example is less abstract than a general proof because specific values are given for the various vectors. On the other hand, because the matrix A is unknown, the questions cannot be answered by direct calculation. The solutions make use of numerical values, but they are not essential for the argument. It is this observation that allows the specific example to serve as representative of a wider class: The same arguments that are used in the example could be used for arbitrary matrices and vectors. The lecturer emphasised this fact in lectures, to his students, on several occasions.

In Figure 2 we show the full solution to Example 3.14. The notes that were available to the students during the lecture contained blank spaces instead of the solutions. Observation 3.5 states that the null space of a matrix has the properties of a subspace.

Example 3.14. Consider an unknown 2×3 matrix A. We know that A satisfies $A\mathbf{x_1} = \mathbf{b_1}$ and $A\mathbf{x_2} = \mathbf{b_2}$, where

$$\mathbf{b_1} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad \mathbf{b_2} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}, \quad \mathbf{x_1} = \begin{pmatrix} 1 \\ 3 \\ -7 \end{pmatrix}, \quad \mathbf{x_2} = \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix}.$$

a. Is $\mathbf{b_1}$ in the range of A? Is $\mathbf{b_2}$ in the range of A?
Solution:
$\mathbf{b_1} \in$ range A because the equation system $A\mathbf{x} = \mathbf{b_1}$ is solvable ($\mathbf{x_1}$ is a solution).
$\mathbf{b_2} \in$ range A because the equation system $A\mathbf{x} = \mathbf{b_2}$ is solvable ($\mathbf{x_2}$ is a solution).

b. Is $\mathbf{b_1} + \mathbf{b_2} = \begin{pmatrix} 1 \\ 8 \end{pmatrix}$ in the range of A?
Solution: Yes. The equation system $A\mathbf{x} = \mathbf{b_1} + \mathbf{b_2}$ is solvable, and $\mathbf{x_1} + \mathbf{x_2} = \begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix}$ is a solution because
$$A(\mathbf{x_1} + \mathbf{x_2}) = A\mathbf{x_1} + A\mathbf{x_2} = \mathbf{b_1} + \mathbf{b_2}.$$

c. Take the number $\lambda = 3$. Is $\lambda\mathbf{b_1} = \begin{pmatrix} 6 \\ 9 \end{pmatrix}$ in the range of A?
Solution: Yes. The equation system $A\mathbf{x} = \lambda\mathbf{b_1}$ is solvable, and $\lambda\mathbf{x_1} = \begin{pmatrix} 3 \\ 4 \\ -21 \end{pmatrix}$ is a solution because
$$A(\lambda\mathbf{x_1}) = \lambda\, A\mathbf{x_1} = \lambda\mathbf{b_1}.$$

d. Is the zero vector **o** in the range of A?
Solution: Yes. The equation system $A\mathbf{x} = \mathbf{o}$ solvable, and $\mathbf{x} = \mathbf{o}$ is a solution because $A\mathbf{o} = \mathbf{o}$.

In this example, we have verified that the range of a matrix has the three properties of Observation 3.5. We can therefore conclude:

Observation 3.15. *The range of a matrix is a subspace.*

Figure 2 The solution of Example 3.14 of the module and the consequential Observation 3.15

Student feedback

Students' views were sought with two questionnaires which highlighted students' preferences and work habits. These were followed by focus group interviews in which Stephanie Treffert-Thomas probed students' views further. As a result the research team learned that students (a) liked the notes-with-gaps, (b) found linear algebra difficult, and (c) focused on learning computations and algorithms rather than engaging with the conceptual understanding as desired by the lecturer. We explain these responses.

(a) Students liked the way that the lecturer had designed the course with the use of notes-with-gaps since they felt it engaged them more. They generally printed the notes and brought them to lectures. One student compared the lecture notes to 'a workbook', and the design of the course as providing a 'stepping stone' from A-level to university. Despite the positive attitude towards the 'gappy' notes this did not necessarily mean that students worked actively on the solution to the examples in lectures. As one student pointed out: "It depended... whether or not I could do it." Students in the focus groups generally acknowledged that many students waited for the solution to be presented by the lecturer, rather than working on it themselves.

(b) Students found linear algebra difficult and particularly challenging at the start. They said that they were unprepared for the conceptual nature of the topic. As one student said, she did not realise "that definitions were important", she was revising from the exercises and examples instead, and realised (too late) that understanding definitions was a requirement for the exams.

(c) Students frequently referred to computational aspects of linear algebra. The Gaussian elimination procedure was taught in the beginning of the module, in Chapter 2.

One student commented that you always had to use Gaussian elimination somewhere at some time, so if she didn't know what to do, she would always do a Gaussian elimination on the matrix. She expressed the view that this was likely to gain at least some marks (in an exam, say).

Synthesis of the teaching approach

We have drawn attention to the informal nature of the teaching approach and its EAG structure. We have also talked about the lecturer's observed levels of commenting. It is important to recall that what we have described is the first semester of the module in which the second semester offers a more formal treatment of the same material; so students are then introduced to vector spaces more generally in a more abstract DTP approach. The first semester is the students' introduction to university mathematics. Thus, the teaching seeks to bridge the school-university transition and prepare students to deal with abstraction.

The EAG approach describes the *structure* of the teaching. Examples are chosen carefully to lead to key concepts through the succeeding argument and generalisation, but without formal proof. The lecturer's commenting is central to this process, offering first a mathematical treatment of the topic in consideration, then a commentary on the relationships involved, emphasising key ideas and ways in which these fit into the broader picture, and finally suggesting to students how they should think about and work on these concepts. Our data showed that students liked the course structure and the course notes. Nevertheless, many students found the transition to argumentation at this level a difficult one, seeking examples which they could follow and taking a more broadly computational approach. Anecdotal evidence from small group tutors suggests that students tackled problem sheets by looking for examples that demonstrated the required approach. Although such responses from students suggest a dependency on the lecturer, a desire for given procedures and a computational approach, towards the end of the year students seemed able to deal with the more abstract treatment, gaining confidence from recognising the material and their earlier struggles with it. They reported that the first semester approach had been valuable in enabling them to address the more abstract formulation in the second semester. A quotation from a focus group shows how two students thought about this.

S1: "I think my understanding of the subject got a bit better and I understand what a lot of the words mean a lot better now [i.e., in Semester 2], so many things like range, basis, then rank, rank-nullity, span, and there are so many of them and try and cram them all in . . .. The way we've used them again and again this term and my small group tutor . . . we've gone over it so many times that I'd be pretty stupid if I didn't get it by now . . . and we went through the class test afterwards in my tutorial and I kind of thought that's really silly, I should have done better."

S2: "Yeah, it did seem very easy afterwards and once we looked at the solutions for it."

In conclusion

Given that students find the transition to abstraction and formalism in university mathematics a difficult one, our research documents an approach which offers an alternative to the traditional DTP. We have shown briefly the key elements of this approach, but in the short space of this article have been able to present only little specific detail and almost no treatment of the ways in which the lecturer's thinking and intentions were realised in the teaching practice and in the responses of students. The latter (intentions and their realisation) is the focus of the PhD thesis of the second author which is forthcoming. In this, Stephanie Treffert-Thomas reports on an *activity theory* analysis of the observational data in order to relate teaching intentions with practical outcomes and link teaching with learning in the mathematical context of linear algebra. We welcome interest in these ideas and invite those interested to get in touch with us for discussion and debate.

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