Problemen

| Problem Section

This Problem Section is open to everyone; everybody is encouraged to send in solutions and propose problems. Group contributions are welcome.
For each problem, the most elegant correct solution will be rewarded with a book token worth 20 euro. At times there will be a Star Problem, to which the proposer does not know any solution. For the first correct solution sent in within one year there is a prize of 100 euro.
When proposing a problem, please either include a complete solution or indicate that it is intended as a Star Problem. Electronic submissions of problems and solutions are preferred (problems@nieuwarchief.nl).
The deadline for solutions to the problems in this edition is December 1st, 2011.

Problem A (Ilya Bogdanov)

Fix a point P in the interior of a face of a regular tetrahedron Δ. Show that Δ can be partitioned in four congruent convex polyhedra such that P is a vertex of one of them.

Problem B (based on a proposal of Benne de Weger)

Let n be a positive integer. Show that 3^n divides the numerator of

$$\sum_{k=1}^{n} \frac{4k-1}{2k(2k-1)} 9^k.$$

Problem C (Marco Golla and Marco Golla)

Let $n > 1$ be an integer. Show that there are no non-linear complex polynomials $f(X)$ such that

$$f^n(X) - X = (f \circ f \circ \cdots \circ f)(X) - X$$

is divisible by $(f(X) - X)^2$.

Edition 2011-1 We have received submissions from Rik Bos (Bunschoten), Pieter de Groen (Brussel), Alex Heinis (Hoofddorp), Thijmen Krebs (Nootdorp), Julian Lyczak (Odijk), Tejaswi Navilarekallu (Amsterdam), Albert Stadler (Herrliberg), Arjen Stolk (Houten), Rohith Varma (Chennai), Rob van der Waall (Huizen) and Martijn Weterings (Wageningen).

Problem 2011-1/A Prove that every commutative ring with identity having at most five ideals is a principal ideal ring.

Solution We have received solutions from Tejaswi Navilarekallu, Julian Lyczak, Arjen Stolk, Rohith Varma and Martijn Weterings. The book token goes to Tejaswi Navilarekallu. The following is based on his solution.
Let R be a commutative ring with at most five ideals. Assume that R is not a principal ideal ring. Then R has a non-principal ideal I of the form $I = (\alpha, \beta)$. It follows that the following five ideals are distinct:

$$0, (\alpha), (\beta), I, R,$$

and hence the ideal $(\alpha + \beta)$ is amongst them. Clearly $(\alpha + \beta)$ cannot be equal to 0 or I. Since $(\alpha + \beta)$ is contained in I, it cannot equal R either, so without loss of generality we may assume that $(\alpha + \beta) = (\alpha)$. But then (α) contains β, hence $(\alpha) = I$, a contradiction.

Problem 2011-1/B Let $(a_n)_{n\geq 1}$ be a sequence of integers that satisfies

$$a_n = a_{n-1} - \min(a_{n-2}, a_{n-3})$$

for all $n \geq 4$. Prove that for every positive integer k there is an n such that a_n is divisible by 3^k.

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Solution We received solutions from Rik Bos, Pieter de Groen, Alex Heinis, Thijmen Krebs, Tejaswi Navilarekallu and Martijn Weterings. The book token goes to Thijmen Krebs.

If there is an $n \in \mathbb{N}$ with $a_n = 0$, the statement is certainly true. So from now on we assume that all a_n are non-zero. Suppose that a_n, a_{n+1}, a_{n+2} are all positive for some $n \in \mathbb{N}$. Then the recurrence shows that the subsequent terms are decreasing until one of them becomes negative. Similarly no infinite sequence of negative terms can occur.

In particular there exists an index $i \geq 3$ such that $a_{i-2} > 0 > a_{i-1}$. Now it follows that $a_{i-3} > a_{i-2}$, hence that $a_i = a_{i-1} - a_{i-2}$ and $a_{i+1} = -a_{i-2} < 0$. In conclusion we have

$$0 > a_{i-1} > a_i \quad \text{and} \quad a_i < a_{i+1} < 0.$$

Denote $x = -a_i$, $y = -a_{i+1}$. We apply the recurrence relation several times:

$$\begin{array}{lll}
a_i = -x, & a_{i+6} = -x + 2y, & a_{i+12} = 2x + 2y - c, \\
a_{i+1} = -y, & a_{i+7} = -2x + y, & a_{i+13} = 3y, \\
a_{i+2} = x - y, & a_{i+8} = -x - y, & a_{i+14} = -3x + 3y, \\
a_{i+3} = 2x - y, & a_{i+9} = x - 2y, & a_{i+15} = -3x, \\
a_{i+4} = 2x, & a_{i+10} = 2x - y - c, & a_{i+16} = -3y, \\
a_{i+5} = x + y, & a_{i+11} = 3x - c, & a_{i+17} = 3x - 3y,
\end{array}$$

where $c = \min(0, 2y - x)$. We find $(a_{i+15}, a_{i+16}, a_{i+17}) = 3(a_i, a_{i+1}, a_{i+2})$. Hence from a_{i+15} on, all terms are divisible by 3. Induction on k shows that from a_{i+15k} on, all terms are divisible by 3^k.

Problem 2011-1/C Find all positive integers N, r, p with p prime that satisfy

$$\prod_{\ell \leq N} \ell = p(p^r + 1),$$

where the product runs over all primes ℓ not exceeding N.

Solution We have received solutions from Alex Heinis, Tejaswi Navilarekallu, Albert Stadler, Rob van der Waall, Julian Lyczak and Martijn Weterings. The book token goes to Alex Heinis. The present solution is based on those of Alex Heinis and Tejaswi Navilarekallu.

We claim that the only triples (N, r, p) that satisfy the requirement are $(3,1,2)$, $(4,1,2)$, $(5,2,3)$, $(6,2,3)$, $(5,1,5)$ and $(6,1,5)$.

We first show that $p \leq 5$. Suppose on the contrary that $p \geq 7$ and hence $N \geq 7$. Note that every prime ℓ not exceeding N divides $p(p^r + 1)$. In particular, if q is a prime that divides $p - 1$ then it also divides $p^r + 1$. But $p^r + 1 \equiv 1 + 1 \equiv 2 \pmod q$. Therefore $p - 1 = 2^k$ for some positive integer k. If k has an odd divisor $d > 1$, then we have $2^{k/d} + 1 \mid 2^k + 1 = p$. This implies that $p = 2^{2^t} + 1$ for some t, and because $p \geq 7$ we may assume $t \geq 2$.

Note that both 3 and 5 divide $p^r + 1$. Divisibility by 3 implies r is odd. On the other hand, we have $p = 2^{2^t} + 1 = 16^{2^{t-2}} + 1 \equiv 2 \pmod 5$, so $p^r + 1 \equiv 2^r + 1 \pmod 5$. Hence, divisibility of $p^r + 1$ by 5 implies that r is even, a contradiction. This proves that $p \leq 5$.

We now treat the cases $p = 2, 3, 5$ separately.

Suppose $p = 2$. Since $r \geq 1$ we have $N \geq 3$. Divisibility of $p^r + 1$ by 3 implies r is odd and hence 5 does not divide $p^r + 1$. Therefore we get $N \leq 4$ in this case.

Suppose $p = 3$. Since $r \geq 1$ we have $N \geq 5$. Since 4 does not divide $3^r + 1$, the integer r is even. This implies that 7 does not divide $3^r + 1$. Therefore $N \leq 6$ in this case.

Suppose $p = 5$. Again we have $N \geq 5$. Now 7 divides $5^r + 1$ if and only if $r \equiv 3 \pmod 6$ if and only if 9 divides $5^r + 1$. By hypothesis, $5^r + 1$ must be square-free, so we get $N \leq 6$.