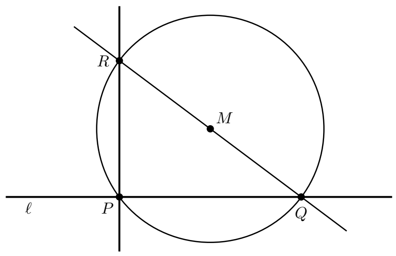
# Problemen

| Problem Section

This Problem Section is open to everyone; everybody is encouraged to send in solutions and propose problems. Group contributions are welcome.
For each problem, the most elegant correct solution will be rewarded with a book token worth 20 euro. At times there will be a Star Problem, to which the proposer does not know any solution. For the first correct solution sent in within one year there is a prize of 100 euro.
When proposing a problem, please either include a complete solution or indicate that it is intended as a Star Problem. Electronic submissions of problems and solutions are preferred (problems@nieuwarchief.nl).
The deadline for solutions to the problems in this edition is December 1st, 2010.

All three problems ask for a construction with ruler and compass in a limited number of steps. More precisely, given a collection of points, lines, and circles in the plane, a *move* consists of adding to the collection either a line through two of the points, or a circle centered at one of them and passing through another. At any time one is allowed to freely add any intersection point among the lines and circles, as well as any sufficiently general point, either in the plane, or on any of the lines or circles.




For example, given a line $\ell$ and a point $P$ on $\ell$ one can construct a line through $P$ and perpendicular to $\ell$ in three moves as follows. Choose a point $M$ not on $\ell$. For the first move, take the circle $C$ centered at $M$ and going through $P$. Let $Q$ be the second point of intersection between $C$ and $\ell$. For the second move, add the line through $Q$ and $M$ and let $R$ be the second point of intersection between this line and $C$. Finally, add the line $PR$, which is perpendicular to $\ell$.
These problems are inspired by Robin Hartshorne's book *Geometry: Euclid and beyond*.

**Problem A**
Given a line and distinct points $A$ and $B$ on it, construct in at most four moves the point $C$ between $A$ and $B$ that satisfies $6|AC| = |AB|$.

**Problem B**
Given a circle, but not its center, construct in at most seven moves an equilateral triangle whose vertices lie on the circle.

**Problem C**
Construct a square in at most eight moves.

**Edition 2010-1** We have received submissions from Thijmen Krebs (Nootdorp), Josephine Buskes (Holthees), Tejaswi Navilarekallu (Amsterdam), Pieter de Groen (Brussel), Shai Covo (Kiryat-Ono), Paolo Perfetti (dip. mat. Rome), Adrian & Ileana Duma (Craiova), Noud Aldenhoven (Nijmegen), Aad Vijn (Den Haag), Dušan Holý & Ladislav Matejíčka (Púchov), and Jan van Neerven (Delft).

**Problem 2010-1/A** Show that for every positive integer $n$ there exists a sequence of $n$ consecutive integers with the property that for every $k$ the $k$-th term can be written as a sum of $k$ distinct squares.

**Solution** This problem was solved by Thijmen Krebs, Josephine Buskes, Tejaswi Navilarekallu, and Pieter de Groen. The following is essentially the solution submitted by

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Thijmen Krebs and Josephine Buskes. The book token goes to Josephine Buskes.
For any integer $a > 1$, the sequence whose first term is the square $4a^{2^n}$ satisfies the requirement. Indeed, repeatedly applying the formula

$$4x^2 + 1 = (2x-1)^2 + 4x,$$

we find

$$4a^{2^n} + 1 = \left(2a^{2^{n-1}} - 1\right)^2 + 4a^{2^{n-1}},$$
$$4a^{2^n} + 2 = \left(2a^{2^{n-1}} - 1\right)^2 + \left(2a^{2^{n-2}} - 1\right)^2 + 4a^{2^{n-2}},$$

etcetera, up to

$$4a^{2^n} + n - 1 = \left(2a^{2^{n-1}} - 1\right)^2 + \left(2a^{2^{n-2}} - 1\right)^2 + \cdots + 4a^2.$$

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**Problem 2010-1/B** The integers of the real line mark positions at which we may place chips. We start with $2n+1$ chips, alternatingly blue and red, at consecutive positions. A *move* is a translation by an integer of a pair of differently coloured chips at adjacent positions to two empty positions, as long as at least one of the new positions is adjacent to one that was already occupied.
Show that it is possible, in a finite sequence of moves, to arrange the chips so that they occupy $2n+1$ consecutive positions again, but now with all blue chips on one side and all red chips on the other. Give upper and lower bounds for the smallest number of moves required.

**Solution** This problem was solved by Pieter de Groen and Tejaswi Navilarekallu. Tejaswi Navilarekallu receives the book token. His solution is shown here.
We shall denote a blue and red chip by 1 and 0 respectively. Thus we start with the string $101010\ldots101$ with the number 1 occurring $n+1$ times. We consider the pairs 10 starting from the left-end. There are $n$ such pairs. We repeat the following $n$ times:

- Move the left-most pair to the right end.

We thus end up with the string $1101010\ldots10$. We now have the $n$ pairs on the right. We continue to move the left-most of these pairs to the right until we have a gap of $n(n-1)$ between the starting 1 and the rest. That is, at the end of this procedure, we will be left with

$$1\ \underbrace{- - - - \cdots - - - -}_{\text{empty gap of length } n(n-1)}\ \underbrace{1010\cdots1010}_{n \text{ pairs}}\ .$$

Note that this takes $n(n-1)/2$ moves.
For $1 \le r \le n$, let $C_r$ denote the configuration

$$1\,\underbrace{- - - - \cdots - - - -}_{\text{empty gap of length } t_r}\underbrace{11\cdots11}_{r\ 1'\text{s}}\underbrace{00\cdots00}_{r\ 0'\text{s}}\underbrace{1010\cdots1010}_{(n-r) \text{ pairs}}$$

with $t_r = n(n-1) - r(r-1)$. Thus we have achieved above the configuration $C_1$ as $t_1 = n(n-1)$. We now shall go from $C_r$ to $C_{r+1}$. We repeat the following $r$ times:

- Move the first instance of 01 (starting from the left) to the right of the left-most 1;
- Move the left-most 10 to the pair of empty slots that was created by the previous move.

For instance, after applying these moves once, we will have

$$1\,\underbrace{- - - - \cdots - - - -}_{\text{empty gap of length } (t_r-2)}\underbrace{11\cdots11}_{r\ 1'\text{s}}\underbrace{00\cdots00}_{(r-1)\ 0'\text{s}}100\underbrace{1010\cdots1010}_{(n-r-1) \text{ pairs}}\ .$$

And after applying these moves $r$ times we will obtain $C_{r+1}$ as $t_{r+1} = t_r - 2r$.

We can thus move from $C_1$ to $C_n$, and in this case $t_n = 0$, which is what we wanted to achieve. This proves that we can get the desired configuration using the allowed moves. Note that it takes $n + n(n-1)/2$ moves to get from the initial configuration to $C_1$ and $2r$ moves to go from $C_r$ to $C_{r+1}$. Therefore, the above procedure takes $n + n(n-1)/2 + n(n-1) = n(3n-1)/2$ moves in total. This gives an upper bound for the smallest number of moves required.
For a lower bound we will give the proposer's solution. Suppose the coordinate of the leftmost 1 is 0 (and thus the coordinate of the rightmost 1 is $2n$). Let $a_0, \ldots, a_n$ be the coordinates of the 1's and let $b_1, \ldots, b_n$ be the coordinates of the 0's. Each move translates a 0 and a 1 over the same distance, which implies that the quantity $\sum_{i=0}^n a_i - \sum_{i=1}^n b_i$ is invariant during the game. In the initial configuration this quantity is $n$ and in the final configuration we have, say, $a_i = c + i$ and $b_i = c + n + i$ so that $\sum_{i=0}^n a_i - \sum_{i=1}^n b_i = c - n^2$, from which $c = n(n+1)$ follows. So the coordinate of the rightmost occupied point in the final configuration is $n(n+1)$ higher than in the initial configuration. Each move raises the coordinate of the rightmost point by at most 2, so $n(n+1)/2$ is a lower bound for the number of moves required.
It thus remains an interesting challenge to improve either of the bounds.

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**Problem 2010-1/C** Is there a function $f\colon \mathbf{R} \to \mathbf{R}$ that is everywhere left continuous but nowhere continuous?

**Solution** The problem was solved by Shai Covo, Paolo Perfetti, Thijmen Krebs, Adrian & Ileana Duma, Noud Aldenhoven, Aad Vijn, Dušan Holý & Ladislav Matejíčka, Tejawsi Navilarekallu, and Jan van Neerven.
The following is based on the solutions submitted by Paolo Perfetti and Shai Covo. The book token goes to Shai Covo.
A function $f : \mathbf{R} \to \mathbf{R}$ that is everywhere left-continuous cannot be nowhere continuous. Indeed, we prove that the set $D$ of discontinuity points of $f$ is at most countable.
We start with a definition.

**Definition** (Left Limit Point). *Let $X$ be a subset of $\mathbf{R}$. A point $y \in \mathbf{R}$ is a* left limit point *of $X$ if for every $\epsilon > 0$ there exists $x \in X$ such that $y - \epsilon < x < y$.*

We also need the following lemma.

**Lemma.** *Let $X$ be an uncountable subset of $\mathbf{R}$. Then there exists a left limit point of $X$.*

*Proof.* Suppose, by contradiction, that every point of $\mathbf{R}$ is not a left limit point of $X$, that is for every point $x \in \mathbf{R}$ there exists $\epsilon_x$ such that the open interval $(x - \epsilon_x, x)$ does not contain points of $X$. All these intervals are disjoint and everyone of them contains at least a rational number. Since there are only countably many rational numbers and the number of intervals is not countable, we are done. ∎

Now suppose that the set $D$ of discountinuity points of $f$ is uncountable and define for every $n \in \mathbf{Z}_{>0}$ the set $D_n$ by

$$D_n = \{x \in \mathbf{R} : \forall \delta > 0\ \exists y \in (x, x+\delta) : |f(y) - f(x)| \geq 1/n\}.$$

Obviously, $D_1 \subseteq D_2 \subseteq D_3 \subseteq \ldots \subseteq D$, and by hypothesis $D = \cup_{n=1}^\infty D_n$. Since a countable union of countable sets is countable, there exists $N \in \mathbf{Z}_{>0}$ such that $D_N$ is uncountable. By the previous lemma there exists a left limit point $l$ of $D_N$, that is we have an increasing sequence $\{x_n\}_{n \geq 1}$ of points of $D_N$ such that $\lim_{n\to\infty} x_n = l$. For every $x_n$ there is $x'_n$ such that $x_n < x'_n < l$ and $|f(x'_n) - f(l)| \geq 1/n$, because $x_n \in D_N$. Then $\lim_{n\to\infty} x'_n = l$, but $|f(x'_n) - f(l)| \geq 1/n$, and this contradicts the fact that $f$ is left-continuous at $l$.
As Dušan Holý & Ladislav Matejíčka pointed out, the answer to this problem is also an immediate consequence of the following theorem.

**Theorem.** *[1] Let $X$, $Y$ be topological spaces and let $X$ be a Baire space. If $f : X \to Y$ is a quasicontinuous function then the set of points of continuity is dense in $X$.*

**References**

[1] T. Neubrunn: *Quasicontinuity*, Real. Anal. Exchange, (Vol. 14, 1988, 259–306).