Problemen

Problem Section

This Problem Section is open to everyone; everybody is encouraged to send in solutions and propose problems. Group contributions are welcome.
For each problem, the most elegant correct solution will be rewarded with a book token worth 20 euro. At times there will be a Star Problem, to which the proposer does not know any solution. For the first correct solution sent in within one year there is a prize of 100 euro.
When proposing a problem, please either include a complete solution or indicate that it is intended as a Star Problem. Electronic submissions of problems and solutions are preferred (problems@nieuwarchief.nl).
The deadline for solutions to the problems in this edition is September 1.

Problem A (proposed by Gabriele Dalla Torre)
Show that for every positive integer n and every integer $m \geq 2$, we have

$$\sum_{\substack{1 \leq i \leq n \\ m \nmid i}} \lfloor \log_m(n/i) \rfloor = \lfloor n/m \rfloor.$$

Problem B (folklore)
Let $b \geq 2$ be an integer. We let $\sigma_b(n)$ denote the sum of the digits in base b of the integer n. Show that we have

$$\lim_{n \to \infty} \sigma_b(n!) = \infty.$$

Problem C (folklore)
Two players play a game of n-in-a-row on an infinite checkerboard. The first player plays with white pieces, the second with black pieces. On each move they place one piece on an empty square. The first player to have n consecutive pieces in a row or column wins. For which values of n is there a winning strategy for one of the players?

Edition 2009-4 We received submissions from Pieter de Groen (Brussel) and Thijmen Krebs (Nootdorp).

Problem 2009-4/A Is there a polynomial with rational coefficients whose minimum on the real line is $\sqrt{2}$?

Solution This problem was solved by Pieter de Groen and Thijmen Krebs. Pieter de Groen receives the book token.
We will show that Krebs' polynomial $f(x) = g(x^2)$ with

$$g(x) = \tfrac{1}{8}(3x^4 - 2x^3 - 12x^2 + 12x + 12)$$

is such a polynomial. Indeed, note that the derivative of g satisfies $4g'(x) = 3(x^2 - 2)(2x-1)$, so that $g(x)$ has local minima at $x = \pm\sqrt{2}$ and a local maximum at $x = \frac{1}{2}$. From $g(0) > g(\sqrt{2})$ we conclude that the minimum of g on the interval $[0, \infty)$ equals $g(\sqrt{2}) = \sqrt{2}$. It follows that the minimum of $f(x) = g(x^2)$ on the real line equals $\sqrt{2}$ as well. ∎

Problem 2009-4/B Are there infinitely many positive integers whose positive divisors sum to a square?

Solution Suppose there are only finitely many such integers and let $N > 1$ be a common multiple. For any $x \in \mathbf{R}$, let $S(x)$ denote the set of all prime powers p^r with $p \leq x$ prime and $r \geq 1$ the smallest integer for which p^r does not divide N.
For every integer n, let $\sigma(n) = \sum_{d|n} d$ be the sum of the divisors of n. The function σ is

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weak multiplicative, meaning that $\sigma(mn) = \sigma(m)\sigma(n)$ whenever m and n are coprime. Let q be any prime larger than $\sigma(m)$ for all $m \in S(N)$ and let t denote the number of primes up to and including q. For all t prime powers $m = p^r \in S(q)$ with p prime, the prime divisors of $\sigma(m)$ are smaller than q; for $p \le N$ this follows by definition of q, while for $p > N$ it follows from the fact that $\sigma(m) = p+1$ is even, so all its prime divisors are at most $(p+1)/2 < q$.
We conclude that the $\mathbf{F}_2$-subspace of $\mathbf{Q}^*/\mathbf{Q}^{*2}$ generated by the elements $\sigma(m)$ for all $m \in S(q)$ is contained in the subspace generated by all primes smaller than q, which has dimension $t-1$. This implies that the t elements $\sigma(m)$ for $m \in S(q)$ are linearly dependent, so there exists a nonempty subset $T \subset S(q)$ such that for $n = \prod_{m \in T} m$ the weak multiplicativity of σ yields $\sigma(n) = \prod_{m\in T} \sigma(m) = 1 \in \mathbf{Q}/\mathbf{Q}^{*2}$. Therefore $\sigma(n)$ is a square, which contradicts the fact that n is not a divisor of N. This proves that there are infinitely many integers whose divisors sum to a square. ■

Problem 2009-4/C For which odd positive integers n do there exist an odd integer $k > n$ and a subset $S \subset \mathbf{Z}/k\mathbf{Z}$ of size n such that for every non-zero element $r \in \mathbf{Z}/k\mathbf{Z}$ the cardinality of the intersection $S \cap (S+r)$ is even? What about even n?

Solution This problem was solved by Thijmen Krebs, who receives the book token. The following solution of the problem is based on his solution.
We claim that for $n \equiv 2 \bmod 4$ and for $n \equiv 3 \bmod 4$ there does not exist any odd integer k with the requested property, whereas for $n \equiv 0 \bmod 4$ and for $n \equiv 1 \bmod 4$ there exists such an odd integer k.
We denote by $\bar{r}$ the residue class of the integer r in $\mathbf{Z}/k\mathbf{Z}$.
Firstly, we suppose $n \equiv 2 \bmod 4$ or $n \equiv 3 \bmod 4$. For any candidates k and S and any $r \in \mathbf{Z} \setminus k\mathbf{Z}$ we have

$$|S \cap (S+\bar{r})| = |S \cap (S-\bar{r})| \equiv 0 \bmod 2.$$

By summing over all $r \in 1, \ldots, k-1$ we get

$$n(n-1) = \sum_{r=1}^{k-1} |S \cap (S+\bar{r})| = \sum_{r=1}^{(k-1)/2} (|S \cap (S+\bar{r})| + |S \cap (S-\bar{r})|) \equiv 0 \bmod 4.$$

This proves the first part of our claim.
Now we will show that for $n \equiv 0 \bmod 4$ and for $n \equiv 1 \bmod 4$ there exists such an odd integer k. Let A be the set $\{0,1,2,4\}$. If $n = 4$ then it is easy to check by hand that for $k = 7$ the subset $\bar{A}$ of $\mathbf{Z}/7\mathbf{Z}$ satisfies the requested property.
If $n \equiv 0 \bmod 4$ we pick any odd integer h greater than $n/4$ and any subset $B \subseteq \{0, \ldots, h-1\}$ of cardinality $n/4$. We claim that we can choose $k = 7h$ and $S = \{ha + b \bmod 7h : a \in A, b \in B\}$. For any $r \in \mathbf{Z}/7h\mathbf{Z}$ we have

$$|S \cap (S+r)| = \sum_{(b,c)\in B\times B} \left| (\overline{hA}) \cap (\overline{hA + r + b - c}) \right| =$$

$$= \sum_{\substack{(b,c)\in B\times B \\ r \equiv c-b \bmod h}} \left| (\overline{hA}) \cap \left(\overline{hA + h\frac{(r+b-c)}{h}} \right) \right|$$

All the terms in the last sum are even and equal to either 2 or 4, because it is equal to what we have computed in the case $n = 4$ and $k = 7$, namely the cardinality of the intersection between the sets $\overline{A}$ and $\overline{A + \frac{(r+b-c)}{h}}$ in $\mathbf{Z}/7\mathbf{Z}$.
Now we suppose $n \equiv 1 \bmod 4$ and let A and B be the sets $\{0, \ldots, (n-1)/2\}$ and $\{1,2\}$, respectively. We claim that we can choose $k = \frac{3(n+1)}{2}$ and $S = \{3a + b \bmod k : a \in A, b \in B\} \setminus \{1 \bmod k\}$. For any $r \in \mathbf{Z} \setminus k\mathbf{Z}$ we have

$$|S \cap (S+\bar{r})| = \sum_{(b,c)\in B\times B} \left| \overline{3A} \cap (\overline{3A + r + b - c}) \right| - \left| S \cap (\overline{1+r}) \right| - \left| S \cap (\overline{1-r}) \right|.$$

Note that the sum $\sum_{(b,c)\in B\times B}\left|\overline{3A}\cap(\overline{3A+r+b-c})\right|$ is equal to $(n+1)$ if $r\equiv 0 \bmod 3$ and to $(n+1)/2$ if $r\not\equiv 0 \bmod 3$. We can conclude by observing that in the first case $\overline{1+r}\in S$ if and only if $\overline{1-r}\in S$ and in the second case $\overline{1+r}\in S$ if and only if $\overline{1-r}\notin S$. ■

Star Problems. In the June 2008 issue, we revisited a selection of unsolved star problems. The first correct solution submitted before July 1, 2009 would earn a book token. In this issue, we publish the last solution that we have received.

Problem (Star) 2008-2/7 For $n=1,2,3,\ldots$ we define the function $\Phi_n\colon \mathbf{R}\to\mathbf{R}$ by

$$\Phi_n(x)=(2n)^x-(2n-1)^x+(2n-2)^x-(2n-3)^x+\cdots+2^x-1.$$

Prove or disprove that for all $x\in\mathbf{R}$ and for all n
1. $\Phi_n'(x)>0$;
2. $\Phi_n''(x)>0$.

What can be said about higher derivatives?

Solution We received an ingenious solution from Juan Arias de Reyna and Jan van de Lune, who are awarded the prize. They show that Φ_n' and Φ_n'' are strictly positive, but that there is an x so that $\Phi_n'''(x)<0$. We limit ourselves to giving a sketch of their solution.

Proof that Φ_n' and Φ_n'' are strictly positive.

First of all, we may restrict ourselves to $x<0$, since for $x\geq 0$ it is clear that all the derivatives of $\Phi_n(x)$ are positive.

Denote the first derivative of $-\Phi_n(-x)$ by ϕ_n and the second derivative of $\Phi_n(-x)$ by ψ_n. We need to show that

$$\phi_n(x)=\frac{\log 2}{2^x}-\frac{\log 3}{3^x}+\cdots-\frac{\log(2n-1)}{(2n-1)^x}+\frac{\log(2n)}{(2n)^x}>0$$

and

$$\psi_n(x)=\frac{(\log 2)^2}{2^x}-\frac{(\log 3)^2}{3^x}+\cdots-\frac{(\log(2n-1))^2}{(2n-1)^x}+\frac{(\log(2n))^2}{(2n)^x}>0$$

for all n and for all $x>0$.

The theory of Dirichlet series gives an entire function $\eta(s)$ so that

$$\lim_{n\to\infty}\phi_n(s)=\eta'(s)$$

and

$$\lim_{n\to\infty}\psi_n(s)=-\eta''(s)$$

for all positive real s. (For $s>1$ this is trivial, by the absolute convergence of the series $\sum_n(-1)^n n^{-s}$.) In fact $\eta(s)=(1-2^{1-s})\zeta(s)$, where $\zeta(s)$ is the Riemann zeta function.

Assume that there exists an n and an $x>0$ so that $\phi_n(x)$ (resp. $\psi_n(x)$) is non-positive. It is not too hard to show that this implies that

$$\eta'(x)\leq 0$$

respectively

$$\eta''(x)\geq 0.$$

It therefore suffices to show that $\eta'(x)>0$ and $\eta''(x)<0$ for all $x>0$.

First, one shows that for all x such that

$$x>2\frac{\log(\log(3))-\log(\log(2))}{\log(3)-\log(2)}\approx 2.2718$$

and all $k>0$, one has

$$\frac{(\log(2k))^2}{(2k)^x} \geq \frac{(\log(2k+1))^2}{(2k+1)^x}.$$

If follows that $\eta''(x)$ is negative for all $x > 2.2718$.
The rest of the argument depends on the following inequality

$$|\eta'''(x)| \leq B(s) := \frac{3+6s}{(1+s)^3} \qquad (x \geq s > 0), \tag{1}$$

the proof of which we postpone.
Now one verifies numerically that

$$\eta''(0) \approx -0.06103 < 0,$$

so that by the maximal slope principle and the inequality (1) one finds

$$\eta''(x) < 0, \text{ for all } x < \frac{-\eta''(0)}{B(0)} \approx 0.020343$$

Repeating this about 20 times one finds $\eta''(x) < 0$ for all x between 0 and 2.28, from which we conclude that $\eta''(x) < 0$ for all positive x, this finishes the proof for the second derivative.
For the first derivative, observe that since $\eta'' < 0$ we have that η' is strictly decreasing. But it is easy to check that $\eta'(x)$ is positive for all x sufficiently large, therefore $\eta'(x) > 0$ for all x.
Proof of (1). We now sketch how to prove the crucial inequality (1).
Let $E\colon \mathbf{R} \to \mathbf{R}$ be the "triangle wave" function of period 2 which satisfies $E(x) = \frac{2x-1}{4}$ for $0 \leq x \leq 1$ and $E(x) = \frac{3-2x}{4}$ for $1 \leq x \leq 2$. One shows that

$$\eta(s) = \frac{1}{2} + \frac{s}{4} + s(s+1)\int_1^\infty \frac{E(x)}{x^{s+2}}dx.$$

Computing the third derivative of this, and using $|E(x)| \leq \frac{1}{4}$ one finds

$$|\eta'''(s)| \leq \frac{3+6s}{(1+s)^3}$$

and the desired inequality follows by noting that the right-hand side is decreasing for $s > 0$.
Higher derivatives. If Φ_n''' were strictly positive for all $x > 0$ then it would follow that $\eta'''(x) \geq 0$ for all $x \geq 0$. But one can verify numerically that

$$\eta'''(0) \approx -0.02347468,$$

a contradiction. ∎