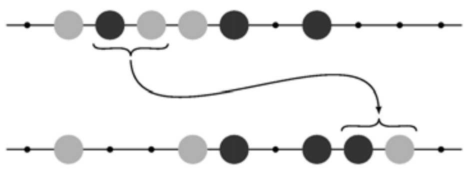
Problemen

| Problem Section

This Problem Section is open to everyone; everybody is encouraged to submit solutions and propose problems. Group contributions are welcome.
For each problem, the most elegant correct solution will be rewarded with a book token worth 20 euro. At times there will be a Star Problem, to which the proposer does not know any solution. For the first correct solution received within one year there is a prize of 100 euro.
When proposing a problem, please either include a complete solution or indicate that it is intended as a Star Problem. Electronic submission of problems and solutions is preferred (problems@nieuwarchief.nl).
The deadline for solutions to problems in this edition is June 1.

Problem A (proposed by Arne Smeets)
Show that for each positive integer n there exists a sequence of n consecutive integers such that for each k, the k-th term can be written as a sum of k distinct squares.

Problem B (proposed by Jos Brakenhoff)
The integers of the real line mark positions at which we may place chips. We start with $2n+1$ chips, alternatingly blue and red, at consecutive positions. A *move* is a translation by an integer of a pair of differently coloured chips at adjacent positions to two empty positions, as long as at least one of the new positions is adjacent to one that was already occupied.



Show that it is possible, in a finite sequence of moves, to arrange the chips so that they occupy $2n+1$ consecutive positions again, but now with all blue chips on one side and all red chips on the other. Give upper and lower bounds for the smallest number of moves required.

Problem C (proposed by Frank Redig)
Does a function $f\colon \mathbf{R} \to \mathbf{R}$ exist that is everywhere left-continuous, but nowhere continuous?

Edition 2009-3 We received submissions from Daniël Worm (Leiden), Rutger Kuyper (Nijmegen), Thijmen Krebs (Nootdorp), Jaap Spies (Emmen), Sander Scholtus (Den Haag), Pieter de Groen (Brussel), Dan Dima (Bucharest), Wim Schikhof (Nijmegen), and Sep Thijssen (Nijmegen).

Problem 2009-3/A Let k be a non-negative integer. Let $S \subset \mathbf{Z}$ be a set consisting of $2^{k+1}-1$ integers. Show there exists a subset $T \subset S$ of cardinality 2^k such that the sum of the elements of T is divisible by 2^k.

Solution This problem was solved by Daniël Worm, Thijmen Krebs, Rutger Kuyper, Sander Scholtus, and Sep Thijssen. All submitted essentially the same solution. The book token goes to Rutger Kuyper.
We use induction. For $k=0$ the statement clearly holds. Suppose the statement holds for some non-negative integer k. Let $S \subset \mathbf{Z}$ be a subset of cardinality $2^{k+2}-1$. Using the induction hypothesis on two disjoints subsets of S of cardinality $2^{k+1}-1$ each, we can find

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disjoint subsets T_1 and T_2 of S of cardinality 2^k, such that the sum of their elements is divisible by 2^k. Now note that the complement of $T_1 \cup T_2$ in S has cardinality $2^{k+1} - 1$, so by using the induction hypothesis once more we find a subset $T_3 \subset S$, disjoint with T_1 and T_2, of cardinality 2^k and such that the sum of its elements is divisible by 2^k. To conclude, choose $i \neq j$ such that T_i and T_j have the same sum modulo 2^{k+1} and observe that $T = T_i \cup T_j \subset S$ satisfies the requirements.

Problem 2009-3/B Find all functions $f : \mathbf{R}_{>0} \to \mathbf{R}_{>0}$ such that

$$f(x+y) \geq f(x) + yf(f(x)) \qquad (1)$$

for all x and y in $\mathbf{R}_{>0}$.

Solution This problem was solved by Dan Dima, Pieter de Groen, Thijmen Krebs, Sep Thijssen, and Daniël Worm. The book token goes to Daniël Worm.
The following is essentially the solution by Thijmen Krebs and Daniël Worm.
Suppose such a function f exists. For all $x, y > 0$ we have $f(x+y) > f(x)$, so f is strictly increasing. For fixed x, the right-hand side of (1) is linear in y, so f is unbounded. Therefore, we may choose an $x > 0$ such that $f(f(x)) > 1$ and a $y > 0$ satisfying $y(f(f(x)) - 1) > x + 1$. Then for $z = x + y$ we have

$$f(z) = f(x+y) \geq f(x) + yf(f(x)) > yf(f(x)) > x + y + 1 = z + 1.$$

However, from $f(z+1) \geq f(z) + f(f(z)) > f(f(z))$ and the fact that f is increasing, we find $z + 1 > f(z)$. From this contradiction we conclude that no such f exists.

Problem 2009-3/C Let V be an infinite-dimensional vector space. Show that the dimension of the dual space V^* equals the cardinality of V^*.

Solution We received no solutions to this problem. Wim Schikhof pointed out that the solution can be found in the literature (G. Köthe, *Topologische Lineare Räume I*, 1960), where it is known as a theorem of Erdős and Kaplansky. Bas Edixhoven communicated the following (folklore) proof.
We denote the cardinality of a set S by $|S|$.
Let V be an infinite-dimensional vector space over a field k. Clearly $|V^*| \geq \dim(V^*)$, so we only need to show $|V^*| \leq \dim(V^*)$.
Choose a basis I of V, using Zorn's Lemma. Let k^I be the set of all functions from I to k and let $k[I]$ be the vector space of all polynomials in the elements of I. Consider the 'evaluation' map

$$e : k^I \to (k[I])^* : f \mapsto \Big[P \mapsto P\big(f(i)_{i \in I}\big)\Big].$$

We claim that the images of e are linearly independent. To see this, let $f_1, \ldots, f_n$ be distinct elements of k^I and let

$$\alpha_1 e(f_1) + \cdots + \alpha_n e(f_n) = 0$$

be a linear relation amongst their images. Note that there is a finite subset $J \subset I$ on which the functions f are already distinct. In particular, for any $1 \leq j \leq n$ we can choose a polynomial $P \in k[J] \subset k[I]$ that evaluates to 1 on f_j and to 0 on all the other f's, so that

$$\alpha_j = \alpha_1 P\big(f_1(i)_{i \in I}\big) + \cdots + \alpha_n P\big(f_n(i)_{i \in I}\big) = 0,$$

which proves the claim.

Oplossingen | Solutions

Now, as the images of e are linearly independent we have

$$|k^I| \leq \dim(k[I]^*).$$

But k^I is isomorphic to V^* and $k[I]$ is isomorphic to V (since I has the same cardinality as the set of all monomials in I), so we conclude $|V^*| \leq (\dim V^*)$.

Problem (Star) 2008-2/1 Let the continuous function $f_1 : (0,1] \to \mathbf{C}$ be such that

$$\int_0^1 f_1(t)dt$$

exists (and is finite) as an improper Riemann integral. Prove that f_1 has a unique extension to $f\colon \mathbf{R}^+ \to \mathbf{C}$ that is

- continuous on $\mathbf{R}^+$,
- differentiable on $(1,\infty)$ and satisfies the differential-difference equation

$$f'(x) = -\frac{1}{x}f(x-1) \qquad (x>1). \tag{2}$$

Also, determine

$$\lim_{x\to\infty} xf(x).$$

Finally, show that, if $\int_0^1 f_1(t)dt = f_1(1)$, then the series $\sum_{n=0}^{\infty} nf(n)$ and the integral

$$\int_0^\infty f(t)dt$$

both converge absolutely and have the same value.

Solution We received solutions from Joris Bierkens and J. Arias de Reyna & J. van de Lune. Joris Bierkens will receive the prize.
The following solution is based on the one given by Bierkens.
Define the functions $f_n\colon (n-1,n] \to \mathbf{C}$ inductively, by

$$f_n(x) := f_{n-1}(n-1) - \int_{n-1}^x \frac{1}{t} f_{n-1}(t-1)dt,$$

and glue them to a function f on $\mathbf{R}^+$. By the properties of the Riemann integral, this f is continuous on $\mathbf{R}^+$ and differentiable on $(1,\infty)$ and it satisfies the differential-difference equation (2). If $g\colon \mathbf{R}^+ \to \mathbf{C}$ is another function with these properties, then we see that $g'(t) = f'(t)$ on $(1,2]$. From $f(1) = g(1)$ we conclude $f = g$ on $(1,2]$. Repeating this argument it follows that $g = f$ everywhere on $\mathbf{R}^+$.
In order to determine $\lim_{x\to\infty} xf(x)$, note that (2) implies

$$(xf(x))' = f(x) - f(x-1). \tag{3}$$

Therefore

$$\lim_{x\to\infty} xf(x) = f(1) + \int_1^\infty (xf(x))'dx = f_1(1) - \int_0^1 f_1(x)dx,$$

provided that the limit $\lim_{x\to\infty} xf(x)$ exists.

For the last part of the problem, integrate (3) to obtain the recursion

$$\int_n^{n+1} f(t)dt = \int_{n-1}^n f(t)dt + (n+1)f(n+1) - nf(n) \qquad (n \geq 1).$$

Now suppose $\int_0^1 f_1(t)dt = f_1(1)$. This recursion implies

$$\int_{n-1}^n f(t)dt = nf(n). \tag{4}$$

We have for $n > 1$

$$\begin{aligned}\int_n^{n+1} |f(t)|dx &= \int_n^{n+1} \left| f(n) - \int_n^x \frac{1}{t} f(t-1)dt \right| dx \\ &\leq |f(n)| + \int_n^{n+1} \frac{1}{n} \int_n^{n+1} |f(t-1)|dtdx \\ &\leq \frac{1}{n}\int_{n-1}^n |f(t)|dt + \frac{1}{n}\int_n^{n+1} |f(t-1)|dt = \frac{2}{n}\int_{n-1}^n |f(t)|dt.\end{aligned}$$

So, by the ratio test, the series

$$\sum_{n=1}^{\infty} \int_{n-1}^n |f(t)|dt$$

converges. Since we have

$$\sum_{n=1}^{\infty} |nf(n)| = \sum_{n=1}^{\infty} \left| \int_{n-1}^n f(t)dt \right| \leq \int_0^{\infty} |f(t)|dt = \sum_{n=1}^{\infty} \int_{n-1}^n |f(t)|dt,$$

we conclude that both $\sum_{n=0}^{\infty} nf(n)$ and $\int_0^{\infty} f(t)dt$ converge absolutely, and from (4) it follows that they have the same limit.

Problem (Star) 2008-2/4 Let $p\colon [0,1] \to \mathbf{R}$ be a continuous function with $p(t) \geq 0$ for all $t \in [0,1]$ and $\int_0^1 p(t)dt = 1$. Does the function $f\colon \mathbf{C} \to \mathbf{C}$ given by

$$f(z) = e^z - \int_0^1 p(t)e^{zt}dt$$

have infinitely many zeroes?

Solution We received solutions from R.A. Kortram and J. Arias de Reyna & J. van de Lune. R.A. Kortram will receive the prize.
The following solution is based on the one given by Kortram.
We shall prove that the answer is 'yes'. The function f has a Taylor series expansion given by

$$f(z) = \sum_{n=1}^{\infty} a_n \frac{z^n}{n!}$$

with $a_n = 1 - \int_0^1 t^n p(t)dt$. The coefficients a_n are real and satisfy $0 < a_n < 1$ so for all $r \in \mathbf{R}_{>0}$ we have

$$M_r(f) := \max_{|z|=r} |f(z)| = f(r) < e^r. \tag{5}$$

This shows that f is of order (at most) 1: the order of the entire function f is the infimum of all m such that $f(z) = O(e^{|z|^m})$ as $z \to \infty$.

From now on, assume that f has only finitely many zeroes $z_1, \dots, z_N$ with multiplicities $e_1, \dots, e_N$. Hadamard's factorization theorem tells us how an entire function of given order can be expressed as product in terms of its zeroes and leads in our case to

$$f(z) = \phi(z)e^{\lambda z + \mu} \quad \text{with} \quad \phi(z) = \prod_{j=1}^{N} (z - z_j)^{e_j}$$

for certain $\lambda, \mu \in \mathbf{C}$. Since the Taylor coefficients of f are real, we have $\phi(z) \in \mathbf{R}[z]$, $\lambda \in \mathbf{R}$ and $e^{\mu} \in \mathbf{R}$ and hence there is a real number c with

$$f(z) = c\phi(z)e^{\lambda z}.$$

Now put $g(z) = \int_0^1 p(t)e^{zt}dt = e^z - f(z)$. We have

$$g(z) = \sum_{n=0}^{\infty} b_n \frac{z^n}{n!} \quad \text{with} \quad b_n = \int_0^1 t^n p(t)dt > 0.$$

Hence for $r \in \mathbf{R}_{>0}$ we have

$$M_r(g) := \max_{|z|=r} |g(z)| = g(r) = e^r - f(r) = e^r - c\phi(r)e^{\lambda r}.$$

The fact that $f(0) = 0$ implies $\deg(\phi) \geq 1$; combining this with (5) we get $\lambda < 1$. So there is an $R \in \mathbf{R}$ such that for all $r > R$ we have $M_r(g) > e^r/2$.

Choose $\varepsilon < 1/4$ and $\delta \in [0, 1)$ with $\int_\delta^1 p(t)dt < \varepsilon$. Then also $\int_\delta^1 t^n p(t)dt < \varepsilon$. Choose K with $\delta^K < \varepsilon$. For $n \geq K$ we have

$$\int_0^\delta t^n p(t)dt \leq \delta^n \int_0^\delta p(t)dt \leq \delta^n \int_0^1 p(t)dt < \varepsilon$$

and thus $b_n < 2\varepsilon$.

For $r \geq R$ we get the following inequality:

$$e^r/2 < M_r(g) = g(r) < \sum_{n=0}^{K-1} b_n \frac{r^n}{n!} + 2\varepsilon \sum_{n=K}^{\infty} \frac{r^n}{n!} < \sum_{n=0}^{K-1} b_n \frac{r^n}{n!} + 2\varepsilon \cdot e^r,$$

which is a contradiction for large r.