Problemen

| Problem Section

This Problem Section is open to everyone; everybody is encouraged to send in solutions and propose problems. Group contributions are welcome.
For each problem, the most elegant correct solution will be rewarded with a book token worth 20 euro. At times there will be a Star Problem, to which the proposer does not know any solution. For the first correct solution sent in within one year there is a prize of 100 euro.
When proposing a problem, please either include a complete solution or indicate that it is intended as a Star Problem. Electronic submissions of problems and solutions are preferred (problems@nieuwarchief.nl).
The deadline for solutions to the problems in this edition is March 1, 2009.

Problem A (proposed by Alexey Kanel)
Is there a polynomial with rational coefficients whose minimum on the real line is $\sqrt{2}$?

Problem B (proposed by Frans Oort)
Are there infinitely many positive integers whose positive divisors sum to a square?

Problem C (proposed by Gabriele Dalla Torre)
For which odd positive integers n do there exist an odd integer $k > n$ and a subset $S \subset \mathbf{Z}/k\mathbf{Z}$ of size n such that for every non-zero element $r \in \mathbf{Z}/k\mathbf{Z}$ the cardinality of the intersection $S \cap (S + r)$ is even? What about even n?

Star problems. In the June 2008 edition of the NAW we revisited a selection of unsolved star problems. Whoever sent in a solution first before July 1, 2009 would receive a book token. In this and upcoming editions we will publish some of the solutions we have received.

Edition 2009-2 We received solutions from Rob van der Waall (Huizen), Thijmen Krebs (Nootdorp), Ruud Jeurissen (Nijmegen), Tejaswi Navilarekallu (Amsterdam), Hendrik Lenstra (Leiden), and Jaap Spies (Emmen).

Problem 2009-2/A In how many ways can one place coins on an $n \times n$ chessboard such that for every square the number of (horizontally or vertically) adjacent squares that contain a coin is odd?

Solution We received solutions from Tejaswi Navilarekallu and Thijmen Krebs. Tejaswi Navilarekallu will receive the book token.
We contend that the required task is impossible if n is odd and that it can be performed in exactly 2^n ways if n is even.
We use integral coordinates (i, j) with $1 \le i, j \le n$ for the squares of the board. We color a square (i, j) white if $i + j$ is even, and black if it is odd. The problems of placing coins on the white and on the black squares are independent. We say that a configuration of coins is legal at a square if that square has an odd number of neighbors containing a coin.
First we treat the case of even n, so $n = 2k$ for some integer k.
Claim: Any configuration of coins on the white half-diagonal $(1, 1)$, $(2, 2)$, …, (k, k) extends to a unique configuration on all the white squares that is legal at all black squares.
Proof. We show by induction on m that any configuration on the half-diagonal extends uniquely to a configuration on the white squares (i, j) with $i + j \le 2m$ that is legal at all black squares (i, j) with $i + j < 2m$. Note that the uniqueness implies that this configuration will be symmetric in the sense that there is a coin on (i, j) if and only if there is a coin on (j, i).
The case $m = 1$ is trivial. For $m \le n$ the induction step is easy, working outwards from

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the half-diagonal. For $m > n$ we work from the squares $(n, 2m-n)$ and $(2m-n, n)$ on the edge towards the square (m, m) on the diagonal. The symmetry guarantees that there is no conflict at the square (m, m). □

It follows that the number of legal configurations on the white squares is 2^k. Of course the same holds for the black squares, so the total number of legal configurations on the board is $2^k \cdot 2^k = 2^n$.

We shall now prove by contradiction that there are no legal configurations if n is odd. Let k be such that $n = 2k+1$. Assume that we are given a legal configuration. Legality at the corner square $(1,1)$ implies that exactly one of $(1,2)$ and $(2,1)$ contains a coin. Legality at $(2,2)$ then implies that either both or none of $(2,3)$ and $(3,2)$ have a coin. Continuing this alternating process we see that for all i either both or none of $(2i, 2i+1)$ and $(2i+1, 2i)$ have a coin. In particular either both or none of $(n-1, n)$ and $(n, n-1)$ have a coin, which contradicts legality at the corner square (n, n).

Problem 2009-2/B A magic $n \times n$ matrix of order r is an $n \times n$ matrix whose entries are non-negative integers and whose row and column sums all equal r. Let $r > 0$ be an integer. Show that a magic $n \times n$ matrix of order r is the sum of r magic $n \times n$ matrices of order 1.

Solution This problem was solved by Ruud Jeurissen, Thijmen Krebs, Tejaswi Navilarekallu, and Jaaps Spies. The following is essentially the solution by Ruud Jeurissen, which was similar to all others. Ruud Jeurissen is the winner of the book token.

We prove the statement by induction on r, the case $r = 1$ being trivial. Suppose M is a magic $n \times n$ matrix of order r. We associate to M the bipartite graph where the two underlying sets R and C of vertices consist of the rows and columns of M respectively, and the i-th row and j-th column are connected by M_{ij} edges. Each subset $S \subset R$ of size k is connected by kr edges to columns in C. Since each column in C has only r edges, this implies that there are at least k columns in C that are connected to S. By Hall's theorem (also known as the marriage theorem), this implies there is a matching from R to C, meaning there is a magic matrix M' of order 1, such that the entries of $M'' = M - M'$ are non-negative. This implies that M'' is a magic matrix of order $r-1$, so by the induction hypothesis M'' is a sum of $r-1$ magic matrices of order 1. We conclude that $M = M'' + M'$ is the sum of r magic matrices of order 1.

Problem 2009-2/C (proposed by Tejaswi Navilarekallu) Find all finite groups G with the property that for all $g, h \in G$ at least one of (g, h), (g, gh) and (h, hg) is a pair of conjugate elements.

Solution We received solutions from Rob van der Waall and Hendrik Lenstra. The proposer and Hendrik Lenstra had similar solutions, and the following is based on both. Hendrik Lenstra will receive the book token.

We claim that the only groups satisfying the given condition are $\{1\}$, $\mathbf{Z}/2\mathbf{Z}$, and the dihedral groups of order 6 and 10.

Clearly the trivial group and the group of order two satisfy the condition, so assume G has $n > 2$ elements.

Let $1 = d_1, d_2, \ldots, d_k$ be the sizes of the conjugacy classes of G. Therefore, $d_1 + \cdots + d_k = n$. Then the set

$$\{(g, h) \in G \times G | g, h \text{ are conjugates}\}$$

has precisely $d_1^2 + d_2^2 + \cdots + d_k^2$ elements. Similary, the sets

$$\{(g, h) \in G \times G | g, gh \text{ are conjugates}\}$$

and

$$\{(g,h) \in G \times G | h, hg \text{ are conjugates}\}$$

have exactly $d_1^2 + \cdots + d_k^2$ elements. Note that $(1,1)$ belongs to all three sets. For G to satisfy the condition in the problem, we need the union of the above three sets to be $G \times G$. In particular, this gives the inequality $3(d_1^2 + \ldots + d_k^2) \geq n^2 + 2$ or equivalently,

$$3(d_2^2 + \cdots + d_k^2) \geq n^2 - 1. \tag{1}$$

Note that the d_i divide n. If for all i we have $d_i \leq n/3$ then

$$3(d_2^2 + \cdots + d_k^2) \leq 3(d_2 + \cdots + d_k)\frac{n}{3} = (n-1)n$$

contradicting the inequality (1). So there must be a conjugacy class C with exactly $n/2$ elements. We are going to show that, except for the identity, all other conjugacy classes have exactly 2 elements.

Since the conjugation action of G on C is transitive the centralizer of an element $c \in C$ has at most two elements, and since that centralizer contains c it follows that $c^2 = 1$, so all elements of C have order 2.

Let a and b be elements of C. We will show by contradiction that ab is not in C. Assume that ab is in C. Then $abab = 1$, so $aba^{-1} = b$, so $a = b$, since the stabilizer of b consists of only 1 and b. We conclude that $ab = 1$, a contradiction.

Next we show that the complement $H = G - C$ is a subgroup. Let x and y in H be given and fix an $r \in C$. By the above x and y can be written as ar and rb respectively, with $a, b \in C$. Then xy equals ab, which is an element of H.

The action by conjugation of an element $c \in C$ on H is an involution, since $c^2 = 1$. The only fixed point of this action is $1 \in H$, because $chc^{-1} = h$ implies $h^{-1}ch = c$ and the centralizer of c is $\{1, c\}$.

Any finite group H with an involution σ that fixes only $1 \in H$ is necessarily abelian, and the involution must be inversion. To see this, first observe that by a counting argument, every element $x \in H$ can be written as $x = \sigma(h)h^{-1}$ for some $h \in H$, then apply σ to obtain $\sigma(x) = x^{-1}$. Hence the automorphism is inversion, and therefore the group is abelian.

So we may assume $d_1 = 1$, $d_2 = n/2$, and $d_3 = \cdots = d_k = 2$. Together with the inequality (1) this implies that $n \leq 10$. It is now easy to check that only the dihedral groups of order 6 and 10 satisfy the required condition (with $n > 2$.)

Problem (Star) 2008-2/3 Let A and B be $n \times n$ matrices over $\mathbf{C}$. Suppose that $\lim_{k\to\infty}(A^k + B^k)$ exists. Show that there exists $M \in \mathbf{C}^{n\times n}$ such that $\lim_{k\to\infty} A^k - kM$ and $\lim_{k\to\infty} B^k + kM$ exist. Give necessary and sufficient conditions on A and B for M to be zero.

Solution This problem was solved by Alex Heinis and Wim Hesselink. As Wim Hesselink sent in a solution first, he will receive the prize. The following is based on both solutions.

Clearly the matrix M is unique, if it exists. For any linear map $f\colon \mathbf{C}^n \to \mathbf{C}^n$, the vector space $\mathbf{C}^n$ is the direct sum of the generalized eigenspaces $E_{f,\lambda} = \ker(f-\lambda)^n$ for eigenvalues λ of f by the theory of Jordan normal forms.

Lemma 1. For any linear map f on $\mathbf{C}^n$ we have $\lim_k f^k = 0$ if and only if every eigenvalue λ of f satisfies $|\lambda| < 1$.

Proof. The only-if part being obvious, we assume that every eigenvalue λ of f satisfies $|\lambda| < 1$. Let λ be such an eigenvalue. Then the restriction f_λ of f to the generalized eigenspace $E_{f,\lambda}$ can be written as $\lambda \cdot \mathrm{id} + m$, with $m^n = 0$. We get $f_\lambda^k = \sum_{j<n} \binom{k}{j} \lambda^{k-j} m^j$, which tends to 0, because in each term $\binom{k}{j}$ only grows polynomially in k. We conclude that f tends to 0. □

Lemma 2. Every eigenvalue λ of A or B satisfies $|\lambda| < 1$ or $\lambda = 1$.

Proof. For two sequences $(X_k)_k$ and $(Y_k)_k$ of matrices we write $X_k \sim Y_k$ if $\lim_k(X_k - Y_k) = 0$. Set $C = \lim_k(A^k + B^k)$. Then we have

$$C - A^{k+1} \sim B^{k+1} \sim B(C - A^k) = BC - BA^k,$$

and therefore $\lim_k (B-A)A^k = (B-I)C$. Let $x \in \mathbf{C}^n$ be an eigenvector for A with eigenvalue λ. Then $\lambda^k(Bx - \lambda x) = (B-A)A^k x$ converges, namely to $(B-I)Cx$. We conclude that either $|\lambda| < 1$ or $\lambda = 1$, or $Bx = \lambda x = Ax$, in which case $2\lambda^k x = (A^k + B^k)x$ converges, and we also find $|\lambda| < 1$ or $\lambda = 1$. The statement for eigenvalues of B follows from symmetry. □

Let a and b denote the linear maps on $\mathbf{C}^n$ defined by multiplication by A and B respectively, whose eigenvalues are given in the previous lemma. Let $p, m, r \colon \mathbf{C}^n \to \mathbf{C}^n$ be the unique linear maps that equal 0, 0, and a, respectively, on the generalized eigenspaces $E_{a,\lambda}$ of a associated to eigenvalues λ with $|\lambda| < 1$, while their restrictions to $E_{a,1}$ equal id, $a - \mathrm{id}$, and 0 respectively. In other words, with respect to the decomposition

$$\mathbf{C}^n \cong E_{a,1} \oplus \left(\bigoplus_{|\lambda|<1} E_{a,\lambda} \right)$$

the maps p, m, r are given as

$$p = \begin{pmatrix} \mathrm{id} & 0 \\ 0 & 0 \end{pmatrix}, \qquad m = \begin{pmatrix} a - \mathrm{id} & 0 \\ 0 & 0 \end{pmatrix}, \qquad r = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}.$$

Then m is nilpotent, we have $a = p + m + r$, and the identities

$$p^2 = p, \quad pm = m = mp, \quad pr = mr = 0 = rm = rp, \quad \lim_k r^k = 0$$

hold, the latter by Lemma 1. Similarly, we may write $b = q + l + s$ where l is nilpotent and

$$q^2 = q, \quad ql = l = lq, \quad qs = ls = 0 = sl = sq, \quad \lim_k s^k = 0.$$

Of course the decomposition in generalised eigenspaces for a and b are not necessarily the same. We have

$$a^k + b^k = r^k + s^k + p + q + \sum_{1 \le j < n} \binom{k}{j} (l^j + m^j).$$

As this has a limit and $\lim_k r^k = \lim_k s^k = 0$, we find that all terms $l^j + m^j$ for $1 \le j \le n$ vanish. In particular, $l + m = 0$ and $l^2 + m^2 = 0$, so that $l = -m$ and $m^2 = 0$. We conclude $a^k = p + r^k + km$ and $b^k = q + s^k - km$, so that $\lim_k a^k - km = p$ and $\lim_k b^k + km = q$. The first statement of the problem follows for the matrix M associated to m. The matrix M is zero if and only if the restriction of a to the generalized eigenspace $E_{a,1}$ is the identity.

Problem (Star) 2008-2/11 Let V be the complex vector space of all functions $f : \mathbf{C} \to \mathbf{C}$. Let W be the smallest linear subspace of V with the properties:

- the function $f(z) = z$ belongs to W,
- for all $f \in W$, $|f| \in W$.

Does $f(z) = \overline{z}$ belong to W?

Solution The following solution is due to David Preiss (Warwick), and was communicated to us by Miklos Laczkovich. Since the solution was already known, there is no prize winner.

We will show that $f(z) = \overline{z}$ does not belong to W. We claim that it suffices to show that there is a complex vector space S of complex valued functions on the circle $\mathbf{R}/2\pi\mathbf{Z}$ with the properties that

- $h(x) = e^{ix}$ belongs to S,
- for all $h \in S$, $|h| \in S$,
- $h(x) = \cos(x)$ does not belong to S.

Indeed, if the function $f(z) = \overline{z}$ is in W, then the function

$$\frac{e^{ix} + f(e^{ix})}{2} = \cos(x)$$

belongs to S.

Construction of S. Let U be the family of regions $U = \{x + iy : \psi(x) < y < \phi(x)\}$, where $\phi, \psi : \mathbf{R} \to \mathbf{R}$ are continuous, $\psi \le 0 < \phi$ and $\{x : \psi(x) = 0\}$ is locally finite in $\mathbf{R}$.
Let H be the set of functions F on $\mathbf{C}$ for which there is a region $U \in$ U so that F is holomorphic on U, and such that there is an $a < 1$ with $\limsup_{z \in U, |z| \to \infty} |F(z)|/|z|^a = 0$.
Let F be the set of continuous functions $f : \mathbf{R}/2\pi\mathbf{Z} \to \mathbf{R}$ with the property that there exist a positive integer n and functions $F_1, \dots, F_n \in$ H (with corresponding regions $U_1, \dots, U_n \in$ U) and open intervals $I_1, \dots, I_n$ covering the circle minus a finite number of points so that $\cos x \in U_j$ and $f(x) = F_j(\cos x)$ for all $1 \le j \le n$ and all $x \in I_j$.
Let S be the set of all functions $\mathbf{R}/2\pi\mathbf{Z} \to \mathbf{C}$ of the form $f + ig + ce^{ix}$ where $f, g \in$ F and $c \in \mathbf{C}$. The set S is a linear subspace of the complex vector space of all complex-valued functions on the circle.
Clearly $h(x) = e^{ix}$ is an element of S.
Proof that S *is closed under* $h \mapsto |h|$. Let h be a function in S, and write h as

$$h(x) = F_j(\cos x) + iG_j(\cos x) + ce^{ix}$$

on the open interval I_j, with $F_j, G_j \in$ H. Let $a_j < 1$ be such that

$$\limsup_{z \in U_j, |z| \to \infty} |F(z)|/|z|^{a_j} = 0 \text{ and } \limsup_{z \in U_j, |z| \to \infty} |G(z)|/|z|^{a_j} = 0.$$

Assuming, as we may, that $\sin x$ does not change sign on any I_j, we have that on each I_j, $|h(x)|^2 = H_j(\cos x)$ where H_j is a linear combination of 1, F_j^2, G_j^2, $F_j(z)z$, $F_j(z)\gamma(z)$, $G_j(z)z$, $G_j(z)\gamma(z)$, where γ is a suitable branch of $\sqrt{1 - z^2}$. Removing from I_j the finite set where $h(x) = 0$ we have that on each remaining interval $|h|$ coincides with a branch of $H_j^{1/2}$ and one verifies that $|h| \in$ F, where the constant can be taken to be $(a + 1)/2$.
Proof that $h(x) = \cos x$ *does not belong to* S. Assume that $\cos x = f(x) + ig(x) + ce^{ix}$ where $c \in \mathbf{C}$ and $f, g \in$ F. Writing $c = u + iv$ and using that f, g are real, we get $f(x) = (1 - u)\cos x + v \sin x$, $g(x) = -v \cos x - u \sin x$. For any interval I on which we can use the definition of $f, g \in$ F (and on which $\sin x \neq 0$) we therefore have $F, G \in$ H and $U \in$ U so that $F(z) = (1 - u)z + v\gamma(z)$ and $G(z) = -vz - u\gamma(z)$, where γ is a branch of $\sqrt{1 - z^2}$ on U. We have that

$$\limsup_{|z| \to \infty, z \in U} \frac{|F(z)|}{|z|} = \limsup_{|z| \to \infty, z \in U} \frac{|((1 - u)z + v\gamma(z))|}{|z|} = \sqrt{(1 - u)^2 + v^2}.$$

Since this limit has to be zero we conclude that $u = 1$ and $v = 0$. A similar argument for G gives that $u = v = 0$, a contradiction.