Problems

| Open Star Problems

Posing mathematical problems is a venerable tradition of the Koninklijk Wiskundig Genootschap. Now and again a star problem illuminates the problem section of the Nieuw Archief voor Wiskunde. The editors do not know any solutions to these problems. Below is a selection of star problems from the last twelve years for which no solutions have been submitted yet.
Whoever first sends in a correct solution before the July 1, 2009 deadline will receive a prize of 50 euros. Solutions can be sent to uwc@nieuwarchief.nl or to the address given below in the left-hand corner; submission by email (in LaTeX) is preferred.
Group contributions are welcome. Participants should repeat their name, address, affiliation and, if applicable, year of study at the beginning of each problem/solution. If you discover a problem has already been solved in the literature please let us know.
The problems and results can also be found on the problem section website www.nieuwarchief.nl/ps.

Problem 1⋆ (proposed by J. van de Lune in NAW, vierde serie deel 14, no. 3, nov. 1996, pp. 429)
Let the continuous function $f_1 : (0,1] \to \mathbf{C}$ be such that

$$\int_0^1 f_1(t)\mathrm{d}t := \lim_{\epsilon\downarrow 0} \int_\epsilon^1 f_1(t)\mathrm{d}t$$

exists (and is finite) as an improper Riemann integral. Prove that f_1 has a unique extension to $f : \mathbf{R}^+ \to \mathbf{C}$ that is
1. continuous on $\mathbf{R}^+$,
2. differentiable on $(1,\infty)$ and satisfies the differential-difference equation

$$f'(x) = -\frac{1}{x} f(x-1) \qquad (x > 1).$$

Also, determine

$$\lim_{x\to\infty} xf(x).$$

Finally, show that, if $\int_0^1 f_1(t)\mathrm{d}t = f_1(1)$, then the series $\sum_{n=1}^\infty nf(n)$ and the integral

$$\int_0^\infty f(t)\mathrm{d}t := \lim_{T\to\infty} \int_0^T f(t)\mathrm{d}t$$

both converge absolutely and have the same value.

Problem 2⋆ (proposed by A. Szilárd-Károly in NAW, vierde serie deel 14, no. 3, nov. 1996, pp. 430)
Let $A_0A_1\ldots A_n$ be an n–simplex and G_i be the centroid of the system formed by the given points except A_i. Denote by A_i' the intersection of the circumscribed hypersphere of the simplex and the line A_iG_i (for $i = 0,\ldots,n$). Find the maximum value of the sum

$$\sum_{i=0}^n \frac{A_iG_i}{A_iA_i'}.$$

Problem 3⋆ (proposed by G.W. Veltkamp in NAW, vierde serie deel 14, no. 3, nov. 1996, pp. 430)
Let A and B be $n\times n$ matrices over $\mathbf{C}$. Suppose that $\lim_{k\to\infty}(A^k + B^k)$ exists. Show that there exists $M \in \mathbf{C}^{\mathbf{n}\times\mathbf{n}}$ such that $\lim_{k\to\infty}(A^k - kM)$ and $\lim_{k\to\infty}(B^k + kM)$ exist. Give necessary and sufficient conditions on A and B for M to be zero.

Section editor: Matthijs Coster
Problem section NAW
Math. Inst. Univ. Leiden
P.O. Box 9512
2300 RA Leiden
uwc@nieuwarchief.nl

Problems

| Open Star Problems

Problem 4⋆ (NAW, vijfde serie deel 2, no. 1, mar. 2001)
Let $p : [0,1] \to \mathbf{R}$ be a continuous function with $p(t) \geq 0$ for all $t \in [0,1]$ and $\int_0^1 p(t)\mathrm{d}t = 1$. Does the integral function $f : \mathbf{C} \to \mathbf{C}$ given by

$$f(z) := e^z - \int_0^1 p(t)e^{zt}\mathrm{d}t$$

have infinitely many zeroes?

Problem 5⋆ (NAW, vijfde serie deel 2, no. 3, sep. 2001)
Consider n $(n \times n)$ matrices with complex coefficients, A_i, such that for all i and j the i^{th} column of A_j is equal to the j^{th} column of A_i. Moreover each $\mathbf{C}$–linear combination of these n matrices is nilpotent (B is called nilpotent if $B^k = 0$ for some $k \leq 1$). Construct an arbitrary $n \times n^2$ matrix by placing the given n matrices one next to the other. Are the rows of this matrix dependent over $\mathbf{C}$?

Problem 6⋆ (NAW, vijfde serie deel 2, no. 3, sep. 2001)
Is there for every natural number N, a natural number k such that the ternary expension of k^2 contains no twos and at least N ones?

Problem 7⋆ (NAW, vijfde serie deel 3, no. 1, mar. 2002)
For $n = 1,2,3,\ldots$ we define the functions $\Phi_n : \mathbf{R} \to \mathbf{R}$ by $\Phi_n(x) = (2n)^x - (2n-1)^x + (2n-2)^x - (2n-3)^x + \ldots + 2^x - 1$. Prove or disprove that for all $x \in \mathbf{R}$ and for all $n = 1,2,3,\ldots$

1. $\Phi'_n(x) > 0$;
2. $\Phi''_n(x) > 0$.

What can be said about higher derivatives?

Problem 8⋆ (NAW, vijfde serie deel 3, no. 2, jul. 2002)
Can you cover an n–dimensional cube by n smaller cubes?

Problem 9⋆ (NAW, vijfde serie deel 3, no. 3, sep. 2002)
Does there exist a continuous surjection $f : [0,1] \to [0,1]^2$ such that every convex set has a convex image?
Reference: Pach and Rogers, *Partly convex Peano curves*, Bull. London Math. Soc. 15 (1983), no. 4, pp. 321–328.

Problem 10⋆ (NAW, vijfde serie deel 3, no. 3, sep. 2002)
Let $\mathbf{x}$ be a vector in $\mathbf{R}^{\mathbf{n}}$ with coordinates in $\{-1,1\}$ each randomly chosen with probability $\frac{1}{2}$. Let $\mathbf{y}$ be a fixed vector of unity in $\mathbf{R}^{\mathbf{n}}$. What is the probability that the inner product $\mathbf{x} \circ \mathbf{y}$ is smaller than 1?

Problem 11⋆ (NAW, vijfde serie deel 4, no. 1, mar. 2003)
Let V be the complex vector space of all functions $f : \mathbf{C} \to \mathbf{C}$. Let W be the smallest linear subspace of V with the properties:
1. the function $f(z) = z$ belongs to W,
2. for all $f \in W$, $|f| \in W$.
Does the function $f(z) = \overline{z}$ belong to W?
Remark: a conjecture in the theory of Boolean algebra can be reduced to this problem.

Problems

| Open Star Problems

Problem 12⋆ (NAW, vijfde serie deel 4, no. 3, sep. 2003)
Determine the maximum area of a rectangle that can be covered by six disks of unit diameter.

Problem 13⋆ (NAW, vijfde serie deel 5, no. 1, mar. 2004)
For $x \in \mathbf{R}$ define

$$P_n(x) := n^n x \left((x+1)^{n+1} - 1\right)^{n-1} - (n+1)^{n-1}\left((x+1)^n - 1\right)^n$$

for $n \geq 2$. Is it true that this polynomial is of the form

$$P_n(x) = \sum_{k=n+2}^{n^2} c_{nk} x^k$$

with $c_{nk} > 0$ for $n+2 \leq k \leq n^2$?

Problem 14⋆ (NAW, vijfde serie deel 6, no. 1, mar. 2005)
Alice and Bob play a game. Alice places $n-1$ candles on a square cake. Bob places an extra candle in the bottom-left corner. Then, for each candle, he cuts a rectangular piece of cake such that the candle is at the bottom-left corner and no other candles are in the rectangle. Bob gets all these n pieces of cake.

1. Is it always possible for Bob to get more than half of the cake?
2. What is the optimal strategy for Alice to hold onto as much cake as possible?

Reference: Ponder This Challenge June 2004 http://domino.research.ibm.com/Comm /wwwr_ponder.nsf/Challenges/June2005.html.

Problem 15⋆ (NAW, vijfde serie deel 7, no. 3, sep. 2006)
Let $f(X) \in \mathbf{Q}[\mathbf{X}]$ be a polynomial of degree n with rational coefficients. Suppose that degree(gcd($f(X), f^{(k)}(X)$)) > 0 for $k = 1 \ldots n-1$. Prove or disprove $f(X) = a(X-b)^n$ for some rational numbers a, b. (See also arXiv:math.AC/0605090, 3 May 2006.)