Chris Budd
FIMA
CMATH
University of Bath
United Kingdom
cjb@maths.bath.ac.uk

Society

Confessions of an industrial mathematician

Mathematical Study Groups with Industry provide since long a unique opportunity for mathematicians, from young to old, from student to professor, to try their math skills at large. The first ones were held in the 1960s in Oxford and soon the phenomenon spread all over the world. The gatherings connect people from industry with academics for the benefit of both. Chris Budd, an enthusiast for over 20 years, tells about this exciting field. This article appeared earlier in the February 2008 issue of Mathematics Today.

What is *industrial mathematics*? Or indeed more generally what is *applied mathematics*? One view, commonly held amongst many 'pure' mathematicians is that applied maths is what you do if you can't do 'real' mathematics and that industrial mathematics is more (or indeed less) of the same only in this case you do it for money. My own view is very different from this (including the money aspect). Applied mathematics is essentially a two way process. It is the business of applying really good mathematics to problems that arise in the real world (or as close an approximation to the real world as you think it is possible to get). It constantly amazes me that this process works at all, and yet it does. Abstract ideas developed for their own sake turn out to have immensely important applications, which is one (but not the only) reason for strongly supporting abstract research. However, just as importantly, applied mathematics is the process of learning *new maths* from problems *motivated by applications*. Anyone who doubts this should ponder how many hugely important mathematical ideas have come from studying applications, varying from calculus and Fourier analysis to nonlinear dynamics and cryptography. It is certainly true that nature has a way of fighting back whenever you try to use maths to understand it, and the better the problem the more that it fights back at you! To solve even seemingly innocuous problems in the real world can often take (and lead to the creation) of very powerful mathematical ideas. Calculus is the perfect example of this. The beauty of the whole business is the way that these same ideas can take on a life of their own, and find applications in fields very different from the one that stimulated them in the first place. This process of applying mathematics in as many ways as possible can change the world. A wonderful example of this is the discovery of radio waves by Maxwell. Here a piece of essentially pure mathematics led to a whole technology which has totally transformed the world in which we live. Imagine a world without TV, radio, mobile phones and the Internet. But that is where we would be without mathematics.

So, how does industrial maths fit into the above? It is still hard to define exactly what industrial maths means, but as far as I'm concerned it's the maths that I do when I work with organisations that are not universities. This certainly includes 'traditional' industry, but (splendidly) it also includes the Met Office, sport, air traffic control, the forensic service, zoos, hospitals, Air Sea rescue organisations, broadcasting companies and local education authorities. I will use the word *industry* to mean all of these and more. Even traditional industry contains a huge variety of different areas ranging from textiles to telecommunications, space to food and from power generation to financial products. What is exciting is that all of these organisations have interesting problems and that a huge number of these problems can be attacked by using a mathematical approach. Indeed, applying the basic principles I described above, not only can the same mathematics be used in many different industries (for example the mathematics of heat transfer is also highly relevant to the finance industry) but by tackling these problems we can learn lots of new maths in the process. It is certainly true that many industrial problems involve routine mathematics and, yes, money is often involved, but this is not the reason that I enjoy working with industry. Much more importantly, tackling industrial problems requires you to *think out of the box* and to take on challenges far removed from traditional topics taught in applied mathematics courses. The result is (hopefully) new mathematics. I think it is fair to say that significant areas of my own research have followed directly from tackling industrial pro-

Figure 1 A scramble crossing in Japan. The behaviour of the pedestrians in this crossing can be analysed by using the theory of discontinuous dynamical systems.

blems. An example of this is my interest in discontinuous dynamics: the study of dynamical systems in which the (usual) smoothness assumption for these systems is removed. Discontinuous dynamics is an immensely rich area of study with many deep mathematical structures such as new types of bifurcation (grazing, border-collisions, corner-collisions), chaotic behaviour and novel routes to chaos such as period-adding. It also has many applications to problems as diverse as impact, friction, switching, rattling, earthquakes, the firing of neurons and the behaviour of crowds of people (see illustration). Studying both the theory and the applications has kept me, my PhD students and numerous collaborators and colleagues busy for many years. However, for me at least, and for many others, the way into discontinuous dynamics came through an industrial application. In my case it was trying to understand the rattling behaviour of loosely fitting boiler tubes. Trying to solve this problem by using the 'usual' theory of smooth dynamical systems quickly ran into trouble as it became apparent that the phenomena that were being observed were quite different from those predicted by the text books. Trying to address this problem immediately forced us to look at non-smooth dynamics (inspired, I should say, by some brilliant theoretical work by the industrialist who was interested in the problem and was delighted to find an academic they could talk to). However the way into this fascinating field could equally well have been a problem in power transmission in a car or the motion of buildings in an earthquake. The point is that an interesting industrial problem, far from just being an excuse to use cheap and dirty maths to make money, has in contrast led to some very exciting new mathematical ideas with many novel applications.

I must confess that I find this constant need to rise to a mathematical challenge to solve an industrial problem intensely stimulating and it continues to act as a driver for much of my research (although I don't neglect the 'pure' aspects of my research as both are needed to be able to do good mathematics). Presently I'll develop this a bit further through a couple of case studies.

How does industrial mathematics work?

Working at the interface of academia and industry is, and continues to be, a constant conflict of interest. Industry, quite rightly, has to concentrate on short term results, obtained against deadlines and may well want the second best answer *tomorrow*, rather than the best answer in a year (or indeed never). The picture I described in the previous section, of the beautiful development of a mathematical theory stimulated by industrial research, may cut little ice to the manager that needs an answer tomorrow. (Indeed, to be quite honest with you, whilst I now know vastly more about discontinuous dynamics than when I first started to look at the problem, I still cannot solve the original question of the boiler tube which turns out to be immensely difficult!). In contrast, the average academic is under intense pressure to publish scholarly research in leading journals, to develop long term research projects and to work with PhD students who often require a lengthy period of training before they are up to (mathematical) speed. This seems, at first sight, to be the exact opposite of the requirements of industry. It can, in fact, be very hard to persuade some of our colleagues on grant review panels that it is worth investing in industrial maths at all; the argument being that if it were any good then industry would pay for it, and if it is not any good then it doesn't deserve a grant! Indeed it gets worse, whilst the life blood of academics is publishing results in the open literature; industry is often constrained (quite reasonably) by the need for strict confidentiality. At first there would appear to be no middle way in which both parties are satisfied. Fortunately however, there are a number of ways forward in which it is possible to satisfy both parties. Perhaps the best of these are the celebrated *Study Groups with Industry* founded in the 1960's by Alan Tayler CBE and John Ockendon FRS. The format of a typical study group is rather like a cross between a learned conference and a paintballing tournament. It lasts a week. On the first day around eight industrial problems are presented by industrialists themselves. Academics then work in teams for a week to try to solve the problems. On the last day the results of their work are presented to an audience of the industrialists and the academics on the other teams. Does this method always lead to a problem being cracked? Sometimes the answer is yes, but this usually means that the problem has limited scientific value. Much more value to both sides are problems that lead both to new ideas and new, and long lasting collaborations, taking both academics and industrialists in completely new directions. The discontinuous dynamics example above was in fact brought to just such a meeting in Edinburgh in 1988. Another vital aspect of the Study Groups is the training that it gives to PhD students (not to mention older academics) in the skills of mathematical modelling, working in teams and of developing effective computer code under severe time pressure. It is remarkable what can be done in the hot house atmosphere of the Study Groups. As a way of stimulating progress in industrial applied mathematics, the Study Groups have no equal. The model which started in Oxford has now been copied all over the world. I have personally been attending such Study Groups (again all round the world) since 1984 and have worked on a remarkable variety of problems including: microwave cooking, land mine detection, overheating fish tanks, fluorescent light tubes, fridges, aircraft fuel tanks, electric arcs, air traffic control, air sea rescue, weather forecasting, boiler tubes and image processing, to name just a few. Of course one week is not long enough to establish a viable collaboration with industry and it is worth considering some other effective ways of linking academia to industry. A key development has been the foundation in 1987 of ECMI, the European Consortium for Mathematics in Industry by a group of active industrial mathematicians including John and Hilary Ockendon, Alan Tayler (Oxford), Sean McKee (Strathclyde) and Helmut Neunzert (Kaiserslautern) amongst others. ECMI has acted to coordinate industrial mathematics across Europe, so that the European Study Groups with Industry (ESGI) take place in different European countries throughout the year, as part of a carefully managed programme of industrial engagement with academia, which also includes regular conferences and reports on success stories. The study groups are supplemented with training camps in which students from all over Europe work together to learn the techniques for mathematical modelling. Indeed the training of PhD students in the hands on techniques to solve industrial mathematics problems is one of the best features of the study groups. The importance of involving students in industrial mathematics cannot be over emphasised, both in terms of the training that they get, and the originality that they bring to the business of solving problems. Indeed, one of my favourite mechanisms for linking academia to industrial problems is through the use of MSc projects. For seven years at Bath we have been running an MSc programme in *Modern Applications* of Mathematics which has been designed to have very close links with industry. All of the students on this course do a short three month project which is often linked to industry, with both an academic and an industrial supervisor. The nice feature about this system is that everyone wins. First and foremost, the students have an interesting project to work on. Secondly we are able to work together with industry on a project with something more closely approximating an industrial timescale and with little real risk of anything seriously going wrong. Thirdly (and to my mind perhaps most importantly) the MSc project can easily lead on to a much more substantial project, such as a PhD project, with the student hitting the ground running at the start. Ideas for such MSc projects may well come from previous Study Groups, from the Industrial Advisory Board of the MSc or (in a recent development) from the splendid Knowledge Transfer Partnership (KTP) in Industrial Applied Mathematics coordinated by the Smith Institute. The KTP is partly funded by the DTI and part by EPSRC and exists to establish, and maintain, links between academia and industry. Similar organisations such as MITACS in Canada or MACSI in the Republic of Ireland, have closely related missions.

Where is industrial mathematics going (or leading us)?

I think it fair to say that some of the great driving forces of 20th Century mathematics have been physics, engineering and latterly biology. This has been a great stimulus for mathematical developments in partial differential equations, dynamical systems, operator theory, functional analysis, numerical analysis, fluid mechanics, solid mechanics, reaction diffusion systems, signal processing and inverse theory to name just a few areas. Many of the problems that have arisen in these fields, especially the 'traditional' industrial mathematics problems (and certainly anything involving fluids or solids) are *continuum problems* described by *deterministic differential equations*. Amongst the variety of techniques that have been used to solve such equations (that is to find out what the answer looks like as opposed to just proving existence and uniqueness) are simple analytical methods such as separation of variables, approximate and formal asymptotic approaches, phase plane analysis, numerical methods for ODEs and PDEs such as finite element or finite volume methods, the calculus of variations and transform methods such as the Fourier and Laplace transforms. (It is worth noting that formal asymptotic methods have long regarded as somewhat second rate methods to be used to find rough answers rather then wait till 'proper' maths did the job correctly. However, stimulated in part by the need to address very challenging applied problems, asymptotic methods have become an area of intense mathematical study, especially the area of exponential asymptotics which looks at problems in which classical asymptotic expansions have to be continued to all orders so that exponentially small — but still very significant — effects can be resolved.) One of the consequences of concentrating on continuous problems described by PDES is that applied mathematics has, for some considerable time, been almost synonymous with fluid or solid mechanics. Whilst these are great subjects of extreme importance, and are central to 'traditional' industries involving problems with heat and mass transfer, they only represent a fraction of the areas that mathematics can be applied to. Here I believe we may see industry driving the agenda of many of possible developments of 21st Century mathematics in a very positive and exciting way. At the risk of making a fool of myself and gazing into the future with too much abandon, I think that the key drivers of mathematics will be problems dealing with *information* (such as genetics, bio-informatics and, of course the growth of the Internet and related systems), problems involving *complexity* in some form (such as problems on many scales with many connecting components and with some form of network describing how the components interact with each other), and problems centred not so much in the traditional industries but in areas such as retail and commerce. To address such problems we must move away from 'traditional' applied mathematics and instead look at the mathematics of discrete systems, systems with huge complexity and systems which are very likely to have a large stochastic component. We will also have to deal with the very difficult issues of how to optimise such systems and to deal with the increasingly large computations that will have to be done on them. One reason that I believe this is that I have seen it happening before my eyes. I have had the privilege (or have been foolish enough) to organise three Study Groups. The first of these, in 1992, had every problem (with one exception) posed in terms of partial differential equations. In contrast the Study Group I organised in Bath in 2006 had ten problems. Of these precisely one involved partial differential equations, one other involved ordinary differential equations. All the rest were a mixture of (dis-

Figure 2 There is lots of maths in chocolate, and it tastes good too!

crete) optimisation, complexity, network theory, discrete geometry, statistics and neural networks. I have seen similar trends in other Study Groups around the world. At a discussion forum at the Bath Study Group the general feeling was one of excitement (mixed with apprehension it must be said) that industry was prepared to bring such challenging problems to be looked at by mathematicians. Personally I greatly welcome this challenge. I also welcome the fact that much of the mathematics that needs to be used and developed to solve these sort of problems is mathematics that has often been thought of as very pure. Two obvious examples are number theory which comes into its own when we have to deal with discrete information (witness the major application of number theory to cryptography) and graph theory which lies at the heart of our understanding of networks. As someone that calls themselves a mathematician (rather than a pure mathematician or an applied mathematician) I strongly welcome these developments. Of course, the new directions that industry is taking applied mathematics pose interesting, and equally challenging, questions about how we should train the 'applied mathematicians' of the future. It is clear to me (at least) that any such training should certainly include discrete mathematics, large scale computing and methods for stochastic problems. How we do this is of course another matter.

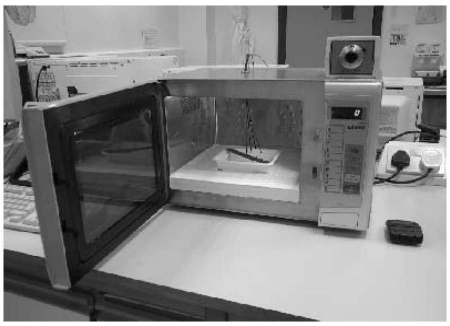

Figure 3 A domestic mode-stirred microwave oven with four temperature probes used to test the predictions of the model

Some case studies

I thought that it would now be appropriate to flesh out the rather general comments above, by looking at a couple of examples. The first is (mainly) an example of a continuum problem whilst the second has a more discrete flavour to it.

Case Study one: Maths can help you to eat

One of the nicest (well certainly it tastes nice) applications of mathematics (well in my opinion at least) arises in the food industry. The food industry takes food from farm to fork, after that it's up to you. Food has to be grown, stored, frozen, defrosted, boiled, transported (possibly when frozen), manufactured, packaged, sorted, marketed, sold to the customer, tested for freshness, cooked, heated, eaten, melted in the mouth (in the case of chocolate) and digested. Nearly all of these stages must be handled very carefully if the food is going to be safe, nutritious and cheap for the customer to eat. It is very easy to think of working on food as a rather trivial application of mathematics, but we must remember that not only is the food industry one of the biggest sources of income to the UK, but also that we all eat food, it affects all of our lives and mistakes in producing food can very quickly make a lot of people very ill. Trivial it is not. Food is also a source of some wonderful mathematical problems, and (a point I take very seriously) the application of maths to food is a splendid way of enthusing young people into the importance of maths in general (especially if you bring free samples along with you!) Some of this maths is very 'traditional' applied maths much of the issues in dealing with food involve classical problems of fluid flow (usually non-Newtonian), solid mechanics (both elastic and visco-elastic), heat transfer, two phase flow, population dynamics (such as fish populations) and free boundary problems. Chocolate manufacture for example, involves very delicate heat flow calculations when manufacturing such delicate items as soft centre chocolates However problems involving the packaging, marketing, distribution and sale of food lie more properly in the realms of optimisation and discrete mathematics. It is clear that the food industry will act as a source of excellent mathematical problems for a very long time to come.

Two problems, in particular, that I have worked on concern the micro-wave cooking and the digestion of food. For the sake of the readers sensitivities I shall only describe the former in any detail, although it is worth saying that modelling digestion is a fascinating exercise in calculating the (chaotic) mixing of nutrients in a highly viscous Non-Newtonian flow driven by pressure gradients and peristaltic motion and with uncertain boundary conditions. As any student knows, a popular way of cooking (or at least of heating) food is to use a micro-wave cooker. In such a cooker, microwaves are generated by a Magnetron (also used in Radar sets) and enter the oven cavity via a waveguide or an antenna. An electric field is then set up inside the oven which irradiates any food placed there. The microwaves penetrate the food and change the orientation of the dipoles in the moist part of the food leading to heating (via friction) of the foodstuff and consequent phase changes.

A problem with this process is that the field can have standing wave patterns, which can result in localised 'cold-spots' where the field is relatively weak. If the food is placed in a cold-spot then its temperature may be lower there and it will be poorly cooked (see figure). To try to avoid this problem the food can either be rotated through the field on a turntable, or the field itself can be 'stirred' by using a rotating metal fan to break up the field patterns.

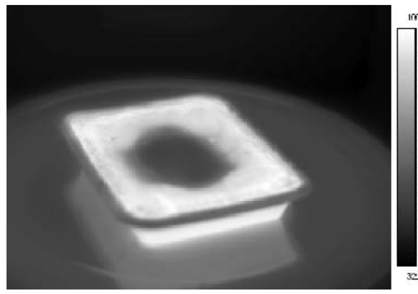

Figure 4 Thermal camera image of food in a domestic turntable oven, showing a distinct 'cold spot' in the centre caused by a local minimum in the radiant electro-magnetic field

An interesting 'industrial mathematics' problem is to model the process by which the food is heated in the oven and to compare the effectiveness of the turntable and mode-stirred designs of the micro-wave oven in heating a moist foodstuff. This problem came to me through the KTN and a Study Group and was 'sub-contracted' to a PhD CASE student Andrew Hill. One way to approach it is to do a full three dimensional field simulation by solving Maxwell's equations, and to then use this to find the temperature by solving the porous medium equations for a two-phase material. The problems with this approach are (i) the computations take a very long time, making it difficult to see the effects of varying the parameters in the problem (ii) it gives little direct insight into the process and the way that it depends upon the parameters and (iii) micro-wave cooking (especially the field distribution) is very sensitive to small changes in the geometry of the cavity, the shape and type of the food and even the humidity of the air. This means that any one calculation may not necessarily give an accurate representation of the electric field of any particular micro-wave oven on any particular day. What is more useful is a representative calculation of the ave-

Figure 5 Three trip-wires are hidden in this image. Can you find them?

rage behaviour of a broad class of (domestic) micro-wave ovens in which the effects of varying the various parameters is more transparent. Here a combination of both an analytical and a numerical calculation proved effective. In this calculation we used a formal asymptotic theory both to calculate an averaged field and to determine how it penetrated inside a moist foodstuff. (Note, contrary to popular myth, microwaves do not cook food from the inside. Instead they penetrate from the outside, and if the food is too large then the interior can receive almost no micro-wave energy, and as a result little direct heating. This is why the manufacturers of micro-wave cookable foods generally insist that, after a period of heating in the oven, the food is stirred to ensure that it is all at a similar temperature.) The temperature T of the food satisfies the equation $u_t = k\nabla^2 T + P(x, y, z, t)$ where u is the enthalpy and P is the power transferred from the microwave field to the food. As remarked above, finding P exactly was very hard, however a good approximation could be found asymptotically (in particular by using the WKBJ method) for ovens with either a mode-stirrer or a turntable. This approximation showed that the overall the field decayed exponentially as it penetrated the food but that on top of this decay was superimposed an oscillatory contribution (due to reflexions of the radiation within the food) the size of which depended upon the dimensions of the food. A relatively simple calculation showed that these oscillations were small provided that the smallest dimension of the food was larger than about 2cm. Fortunately, most foodstuffs satisfy this condition. As a result it was possible to use a much simpler description of the electric field in the enthalpy equation than that given by a full solution of Maxwell's equations, and numerical approximations to the solution of these simplified equations were found very quickly on a desktop PC. When compared against experimental values of both the temperature and the moisture content these solutions were surprisingly accurate, given the approximations that are made, and gave confidence in the use of the model for further design calculations. It was the combination of mathematics, numerical methods, physical modelling and the careful use of experimental data that made this whole approach successful and is typical of the mix of ideas that have to be combined to do effective industrial mathematics.

Case Study Two: Maths Can Save Your Life

One of my favourite 'industrial mathematics' problems came up in a recent study group, and is an example of an application of signal processing and information theory which can potentially save peoples lives. One of the nastiest aspects of the modern world is the existence of anti-personnel land mines. These unpleasant devices, when detonated, jump up into the air and kill anyone close by. They are typically triggered by trip-wires which are attached to the detonators. If someone catches their foot on a trip wire then the mine is detonated and the person dies. To make things a lot worse, the land mines are typically hidden in dense foliage and thin nylon fishing line is used to make the trip wires almost invisible. One way to detect the land mines is to look for the trip wires them selves. However, the foliage either hides the trip-wires, or leaf stems can even resemble a trip wire. Any detection algorithm must work quickly, detect trip-wires

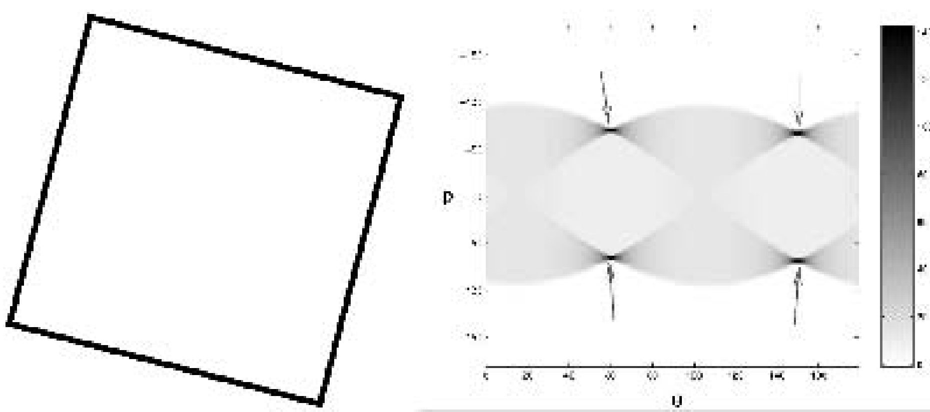

Figure 6 Left: a square; right: the Radon Transform of the square on the left. The four straight lines that make up the sides of the square show up as points of high density.

when they exist and not get confused by finding leaves. An example of the problem that such an algorithm has to face is given in figure 5 in which some trip-wires are hidden in an artificial jungle.

In order to detect the trip wires we must find a way of finding partly obscured straight lines in an image. Fortunately, just such a method exists; it is the Radon Transform (or its various discrete versions). In this transform, the line integral $R(\theta,\rho)$ of an image with intensity $u(x,y)$ is computed along a line at an angle θ and a distance ρ from the centre of the image so that

$$R(\theta,\rho) = \int u(\rho\cos(\theta) - s\sin(\theta), \\ \rho\sin(\theta) + s\cos(\theta))\,ds.$$

This transformation lies at the heart of the CAT scanners used in medical image processing and other applications as it is closely related to the formula for the attenuation of an X-Ray (which is long and straight, like a trip-wire) as it passes through a medium of variable density (such as a human body). Indeed, finding (quickly) the inverse to the Radon Transform of a (potentially noisy) image is one of the key problems of modern image processing. It has countless applications, from detecting tumours in the brain of a patient (and hence saving your life that way), to finding out what (or who) killed King Tutankhamen. For the problem of finding the trip-wires we don't need to find the inverse, instead we can apply the Radon transform directly to the image. In figure 6 see on the left a square and on the right its Radon Transform.

The key point to note in these two images is that the four straight lines making up the sides of the square show up as points of high intensity (arrowed) in the Radon Transform and we can easily read off their orientations. Basically the Radon Transform is good at finding straight lines which is just what we need to detect the trip-wires. Of course life isn't quite as simple as this for real images of trip-wires and some extra work has to be done to detect them. In order to apply the Radon transform the image must first be pre-processed (using a Laplacian filter and an edge detector) to enhance any edges. Following the application of the transform to the enhanced image a threshold must then applied to the resulting values to distinguish between true straight lines caused by trip wires (corresponding to large values of R) and false lines caused by short leaf stems (for which R is not quite as large). However, following a sequence of calibration calculations and analytical estimates with a number of different images, it was possible to derive a fast algorithm which detected the trip-wires by first filtering the image, then applying the Radon Transform, then applying a threshold and then applying the inverse Radon Transform. (The beauty of this is that most of these algorithms are present in the MATLAB Signal Processing Toolbox. Indeed, I consider MATLAB to be one of the greatest tools available to the industrial mathematician.) The result of applying this method to the previous image is given in figure 7 in which the three detected trip-wires are highlighted.

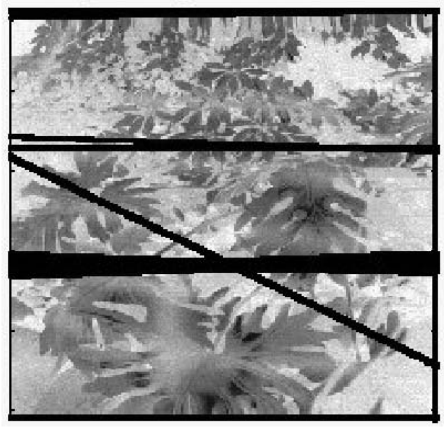

Figure 7 The three trip wires detected using the Radon Transform

Note how the method has not only detected the trip-wires, but, from the width of the lines, an indication is given of the reliability of the calculation. All in all this problem is a very nice combination of analysis and computation.

Signal processing problems of this form are not typically taught in a typical applied mathematics undergraduate course. This is not only a shame, but denies the students on those courses the opportunity to see a major application of mathematics to modern technology.

Conclusions

I hope that I have managed to convey some of the flavour of industrial mathematics as I see it. Far from being a subject of limited academic value, only done for money, industrial maths presents a vibrant intellectual challenge with limitless opportunities for growth and development. This poses significant challenges for the future, not least in the way that we train the next generation of students to prepare them for the very exciting ways that maths will be applied in the future, and the new maths that we will learn from these applications.