Alexander Schrijver
CWI and University of Amsterdam
Kruislaan 413
1098 SJ Amsterdam, Netherlands
lex@cwi.nl

Research

Flows in railway optimization

The Netherlands has an intricate network of railway tracks to connect its many closely situated cities. Recently, the railroad timetable has been completely updated. Unlike the yearly manual adjustments of the past, this update was done using mathematical techniques. For the Dutch public, this was a rare opportunity to come into contact with mathematics. The research that resulted in the new timetable won a team of CWI consisting, besides Lex Schrijver, of Gábor Maróti and Adri Steenbeek the prestiguous Franz Edelman Award for Achievement in Operations Research and the Management Sciences.

In December 2006, *Nederlandse Spoorwegen* (NS, Dutch Railways) introduced the *Spoorboekje 2007* (Timetable 2007). It included a new structure of the train timetable and new plans for train circulation and crew scheduling. Several new connections were made and train lengths were better matched to the number of passengers, while on the other hand train times were scheduled less tightly, a number of transfers were cancelled and some direct train connections were cut into parts in order to reduce the propagation of delays.

Mathematicians were involved in designing the algorithms that created the new schedules. Until 2007, new timetables were made by 'manually' adapting the timetable introduced in 1975. The increase in passenger numbers and the addition of extra infrastructure (new lines and forks, four tracks, and fly-overs) was met mainly by shifting train times and inserting new trains between existing trains. The need for a completely new timetable arose since hardly any trains could be added to the existing schedule where needed. The schedulers at NS felt they needed algorithms to better exploit the capacity of the railway network, and that is why NS asked mathematicians.

While the Netherlands is not large, it is one of the most densely populated countries of the world. The Dutch railway network is correspondingly rather dense, with many short trajectories, run by frequent trains, having several transfer connections on the way. Moreover, space to extend infrastructure is limited in the Netherlands.

These characteristics mean that railway optimization in the Netherlands is faced with rather specific problems compared to most other countries and no adequate off-the-shelf software was therefore available. The algorithms for timetabling and train circulation were made at the Center for Mathematics and Computer Science (CWI) in Amsterdam, for crew scheduling at the University of Padova, and for routing trains through stations at the Erasmus University in Rotterdam.

As might have been expected, the changes raised much public discussion. In an editorial commentary, the Dutch nationwide newspaper *NRC Handelsblad* characterized the new timetable as "the only form of higher mathematics that arouses furious emotions in the country". In a reaction, the CEO of NS claimed that the new timetable was the best possible given the existing infrastructure.

Journalists phoned to ask me if indeed mathematically no better schedule was possible. My answer amounted to the fact that optimality depends on the boundary conditions, that is, in this case, the input of the algorithm: which trains and connections do you want to have, how often, where do trains stop, etc. This is not a question of mathematics but a question of company policy, and not only passenger comfort plays a role here but also other objectives and constraints like crew scheduling, profit, punctuality and infrastructure. Given these constraints and these choices made by the company, the algorithm searches for an optimum timetable.

Economically, the new timetable appears to be successful. The yearly profit of NS is projected to increase by 70 million euros due to the new schedules. On trajectories with the largest timetable improvements, the number of passengers has increased by 10–15%. The effect of this on societal economics and welfare has been estimated tentatively at hundreds of millions of euros. Moreover, despite more trains, the punctuality has increased and the economical effect of this is of the order of tens of millions of euros.

As mentioned, *Timetable 2007* means not only the introduction of a new timetable but also of new systems for rolling stock circulation, to increase seat availability for passengers, for crew scheduling, to improve train personnel rosters, and for routing trains through stations. At CWI we made algorithms for timetabling and for rolling stock circulation. Although the timetabling problem (in particular finding the cyclic, hourly pattern) is also interesting from a network point of view, in this article we just focus on rolling stock circulation, also because of its historical interest.

Transportation and circulation problems belong to the classical problems in operations research and are motivated by application to railways. Flow techniques are basic to it and we first describe two results that are of historical interest. After that, we get back to railway circulation at NS.

Cargo transportation in the Soviet Union

The earliest attributions to the mathematical study of transportation problems are usually to around 1940. The transportation problem was formulated by Hitchcock [4] and a cycle criterion for optimality was considered around the same time independently by Kantorovich in the Soviet Union and Koopmans in the USA. For political reasons, they published their results only later ([5,6]). When Kantorovich found his method, publishing about economics in the Soviet Union was very risky, as it was a politicized issue.

Koopmans, a Dutch mathematical economist who fled to the USA at the beginning of the Second World War, found his methods when appointed at the Combined Shipping Adjustment Board, a British-American agency that routed merchant ships during the Second World War, as they had to sail in convoys under military protection. In 1975, Kantorovich and Koopmans received the Nobel Prize in Economics for their work on transportation.

Less known is that earlier, in 1930, A.N. Tolstoĭ [7] found methods similar to those of Kantorovich and Koopmans. His article called *Methods of finding the minimal total kilometrage in cargo-transportation planning in space* was published in a book on transportation planning issued by the National Commissariat of Transportation of the Soviet Union.

Tolstoĭ presented a number of approaches for the transportation problem, including the now well-known idea that an optimum solution does not have any negative-cost cycle in its residual graph. This residual graph is obtained from the network by adding, to any arc on which the flow is positive, an arc in the reverse direction, with cost equal to the negative of the cost of the forward arc. Here 'cost' can be true cost, length or anything similar.

Tolstoĭ seems to be the first to observe that this cycle condition is necessary for optimality. Moreover, he assumed, but did not explicitly state or prove, the fact that checking the cycle condition is also sufficient for optimality.

Tolstoĭ illuminated his approach with applications to the transportation of salt, cement and other cargo between sources and destinations along the railway network of the Soviet Union. In the paper, he explains the solution of a concrete cargo transportation problem along the Soviet railway network. It has 10 sources and 68 destinations, and 155 links between sources and destinations, and is therefore quite large-scale for that time.

Tolstoĭ 'verifies' the solution by considering a number of cycles in the network and he concludes that his solution is optimum:

"Thus, by use of successive applications of the method of differences, followed by a verification of the results by the circle dependency, we managed to compose the transportation plan which results in the minimum total kilometrage."

Checking Tolstoĭ's problem with modern linear programming tools shows that his solution is indeed optimum.

Max-flow min-cut

The Soviet rail system also aroused the interest of the Americans and again it inspired fundamental research in optimization.

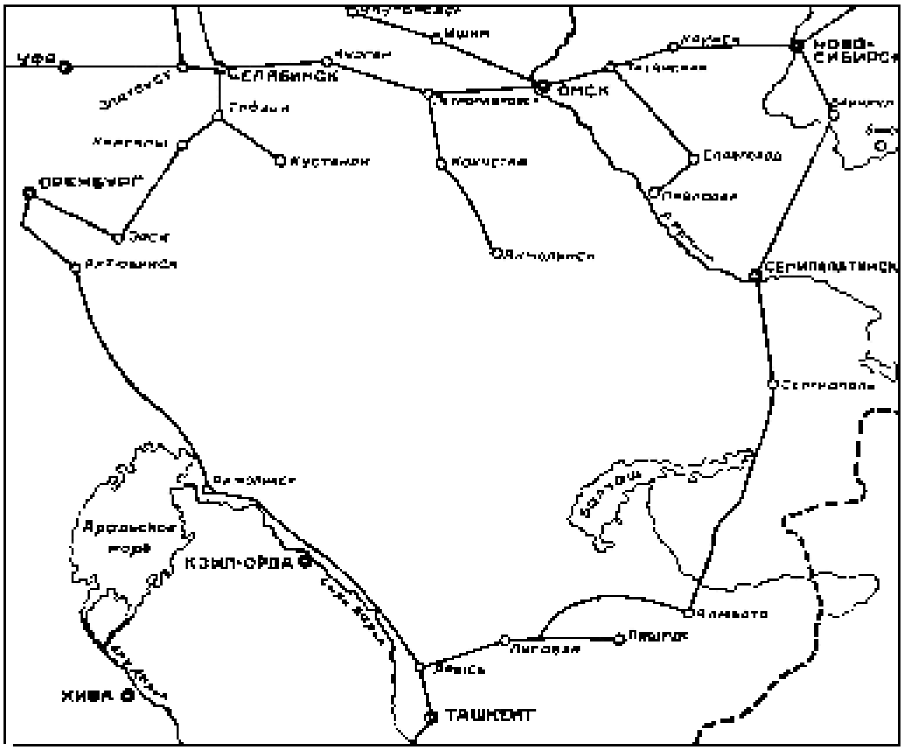

Figure 1 Figure from Tolstoĭ [7] to illustrate a negative cycle

In their basic paper *Maximal Flow through a Network* (published in 1954), Ford and Fulkerson [1] mention that the maximum flow problem was formulated to them by T.E. Harris as follows:

"Consider a rail network connecting two cities by way of a number of intermediate cities, where each link of the network has a number assigned to it representing its capacity. Assuming a steady state condition, find a maximal flow from one given city to the other."

It inspired Ford and Fulkerson to their famous Max-Flow Min-Cut Theorem: *The maximum amount of flow that can be sent along a network from a set of sources to a set of destinations, subject to a given capacity upper bound, is equal to the minimum capacity of the cuts of the network that separate all sources from all destinations.*

In their 1962 book *Flows in Networks*, Ford and Fulkerson [2] give a more precise reference to the origin of the problem:

"It was posed to the authors in the spring of 1955 by T. E. Harris, who, in conjunction with General F. S. Ross (Ret.), had formulated a simplified model of railway traffic flow, and pinpointed this particular problem as the central one suggested by the model [11]."

Ford-Fulkerson's reference 11 is a secret report by Harris and Ross [3] entitled *Fundamentals of a Method for Evaluating Rail Net Capacities*, dated 24 October 1955 and written for the US Air Force. At our request, the Pentagon downgraded it to 'unclassified' on 21 May 1999.

In fact, the Harris-Ross report solves a relatively large-scale maximum flow problem coming from the railway network in the Western Soviet Union and Eastern Europe ('satellite countries'). And the interest of Harris and Ross was not to find a maximum flow but rather a minimum cut ('interdiction') of the Soviet railway system. (Recall that the report was written for the Air Force.) We quote:

"Air power is an effective means of interdicting an enemy's rail system, and such usage is a logical and important mission for this Arm. As in many military operations, however, the success of interdiction depends largely on how complete, accurate, and timely is the commander's information, particularly concerning the effect of his interdiction-program efforts on the enemy's capability to move men and supplies. This information should be available at the time the results are being achieved.

The present paper describes the fundamentals of a method intended to help the specialist who is engaged in estimating railway capabilities, so that he might more readily accomplish this purpose and thus assist the commander and his staff with greater efficiency than is possible at present."

The Harris-Ross report stresses that specialists remain needed to make up the model (which is always a good tactic to get new methods accepted):

"The ability to estimate with relative accuracy the capacity of single railway lines is largely an art. Specialists in this field have no authoritative text (insofar as the authors are informed) to guide their efforts, and very few individuals have either the experience or talent for this type of work. The authors assume that this job will continue to be done by the specialist."'

Whereas experts are needed to set up the model, to solve it is routine (when having the 'work sheets', which were added to the report).

The Harris-Ross report describes an application to the Soviet and East European railways. For the data it refers to several secret reports of the Central Intelligence Agency (CIA) on sections of the Soviet and East European railway networks. After the aggregation of railway divisions to vertices, the network has 44 vertices and 105 (undirected) edges.

The report applies flow techniques to obtain a maximum flow from sources in the Soviet Union to destinations in East European 'satellite' countries (Poland, Czechoslovakia, Austria and East Germany) but the main objective was to find the corresponding minimum cut separating the sources from the destinations. In the report, the minimum cut is indicated as 'The bottleneck' (see Figure 4).

While Tolstoĭ and Harris-Ross had the same railway network as object, their objectives were dual.

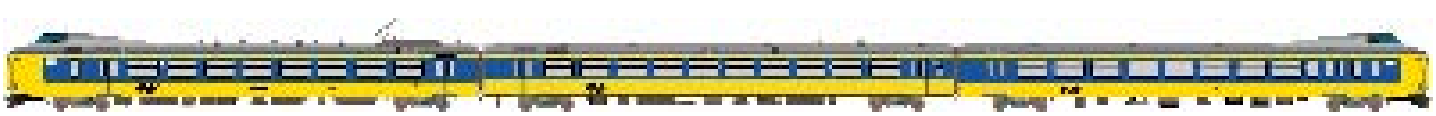

Figure 2 A 'koploper'

train number	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
Amsterdam V		[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
Rotterdam A		7.55	8.55	9.55	10.55	11.55	12.55	13.55	14.55	15.55	16.55	17.55	18.55	19.55	20.55	21.55	22.55	23.55
Rotterdam V	7.00	8.02	9.02	10.02	11.02	12.02	13.02	14.02	15.02	16.02	17.02	18.01	19.02	20.02	21.02	22.02	23.02	
Roosendaal A	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	
Roosendaal V	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]		
Vlissingen A	8.36	9.36	10.36	11.36	12.36	[illegible]	[illegible]	[illegible]	16.36	[illegible]	[illegible]	[illegible]	20.36	21.36	22.36	[illegible]		

train number	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
Vlissingen V			[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	
Roosendaal A			[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	
Roosendaal V		[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
Rotterdam A		[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
Rotterdam V	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
Amsterdam A	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

Figure 3 Timetable Amsterdam-Vlissingen and Vlissingen-Amsterdam

Rolling stock circulation in the Netherlands

We will finally describe a more recent (and more peaceful) application of flow methods to railways, as used by Nederlandse Spoorwegen for *Timetable 2007*.

NS runs an hourly train service on its route Amsterdam-Rotterdam-Roosendaal-Vlissingen and vice versa, with the timetable shown above.

The trains have more stops but for our purposes only those given in the table are of interest since at the stations given train sections can be coupled or separated. For each of the stages of any scheduled train, NS has estimated the number of passengers, as given in the table on the next page (all data concerns weekdays and 2nd class seats).

The problem to be solved is:

What is the minimum amount of train stock necessary to perform this train service in such a way that at each stage there are enough seats?

In order to answer this question, one should know a number of further characteristics and constraints. In a first version of the problem considered, the train stock consisted of one type of two-way train units ('koplopers'), each consisting of three carriages. Each unit has 163 seats.

Each unit has at both ends an engineer's cabin and units can be coupled together up to a certain maximum length (often 15 carriages, meaning in this case 5 train units).

The train length can be changed, by coupling or decoupling units, at the terminal stations of the line, that is at Amsterdam and Vlissingen and *en route* at the intermediate stations Rotterdam and Roosendaal. Any train unit decoupled from a train arriving at place p at time t can be linked up to any other train departing from p at any time later than t (the Amsterdam-Vlissingen schedule is such that in practice this gives enough time to make the necessary switchings).

A last condition is that for each place $p \in$ {Amsterdam, Rotterdam, Roosendaal, Vlissingen}, the number of train units staying overnight at p should be constant during the week (but may vary for different places). This requirement is made to facilitate surveying the stock and to equalize at any place the load of overnight cleaning and maintenance throughout the week. It is not required that the same train unit, after a night in Roosendaal, for example, should return to Roosendaal at the end of the day. Only the number of units is of importance.

Given these problem data and characteristics, one may ask for the minimum number of train units that should be available to perform the daily cycle of train rides required.

It is assumed that if there is sufficient stock for Monday till Friday then this should also be enough for the weekend services since at the weekend a few early trains are cancelled and on the remaining trains there is a smaller expected number of passengers. Moreover, it is not taken into consideration that stock can be exchanged during the day with other lines of the network. In practice this will happen but initially this possibility is ignored.

A network model

If only one type of railway stock is used, clas-

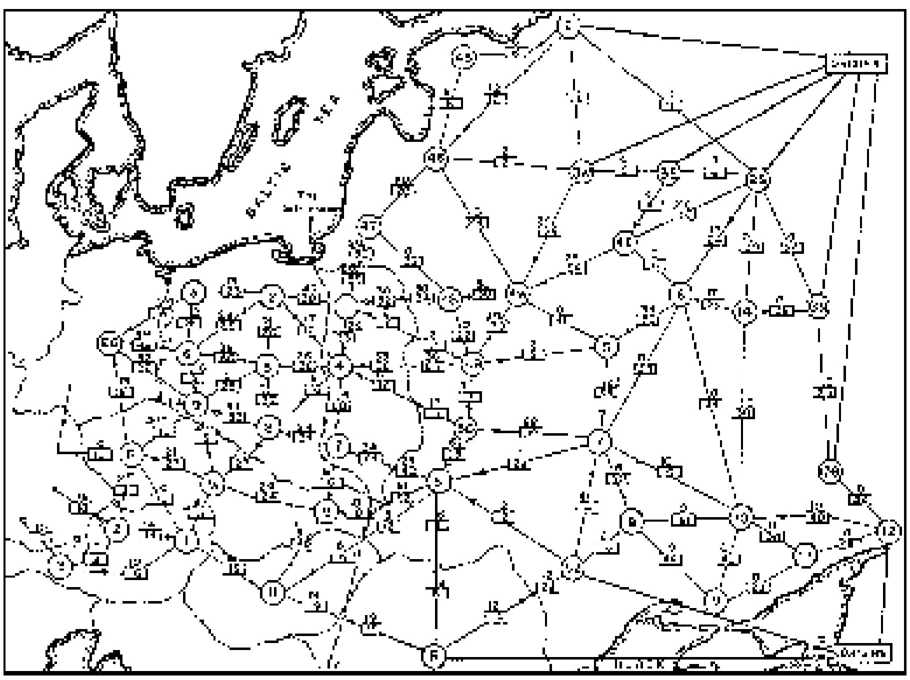

Figure 4 From Harris and Ross [3]: Schematic diagram of the railway network of the Western Soviet Union and East European countries, with a maximum flow of value 163,000 tons from Russia to Eastern Europe and a cut of capacity 163,000 tons indicated as 'The bottleneck'

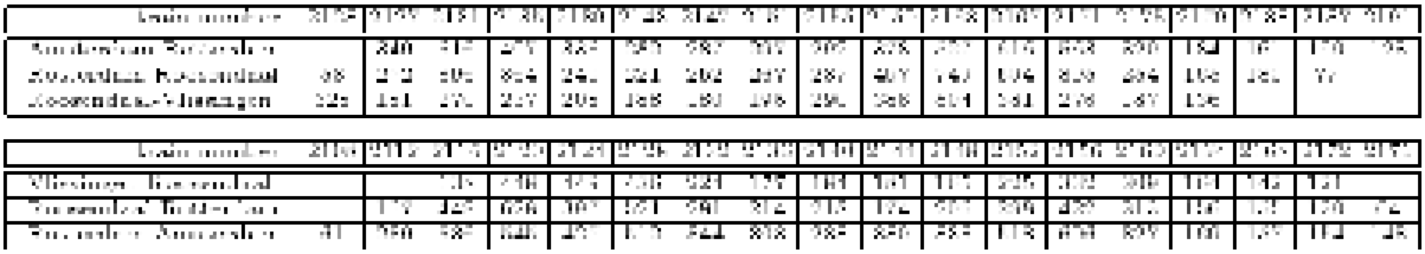

Figure 5 Number of required seats

sical minimum-cost flow techniques (rooted in the work of Tolstoĭ) can be applied. To this end, a directed graph $G = (V, A)$ is constructed as follows. For each place $p \in$ {Amsterdam, Rotterdam, Roosendaal, Vlissingen} and for each time t at which any train leaves or arrives at p, we make a vertex (p, t). So the vertices of G correspond to all 198 time entries in the timetable (Figure 3).

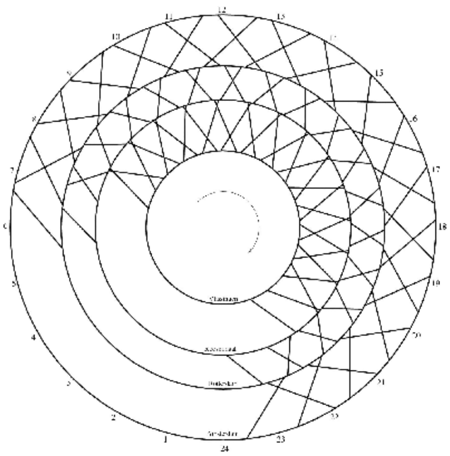

Figure 6 The graph G. All arcs are oriented clockwise.

For any stage of any train ride, leaving place p at time t and arriving at place q at time t', we make a directed arc from (p, t) to (q, t'). For instance, there is an arc from (Roosendaal, 7.43) to (Vlissingen, 8.38).

Moreover, for any place p and any two successive times t, t' at which any train leaves or arrives at p, we make an arc from (p, t) to (p, t'). Thus in our example there will be arcs for example from (Rotterdam, 8.01) to (Rotterdam, 8.32) and from (Rotterdam, 8.32) to (Rotterdam, 8.35).

Finally, for each place p there will be an arc from (p, t) to (p, t'), where t is the last time of the day at which any train leaves or arrives at p and where t' is the first time of the day at which any train leaves or arrives at p. So there is an arc from (Roosendaal, 23.54) to (Roosendaal, 5.29).

We can now describe any possible routing of train stock as a function $f : A \longrightarrow \mathbf{Z}_+$, where for any arc a, $f(a)$ denotes the following. If a corresponds to a ride stage, then $f(a)$ is the number of units deployed for that stage. So if a is the arc from (Roosendaal, 7.43) to (Vlissingen, 8.38), then $f(a) = 4$ means that 4 coupled train units will run this train stage. If a corresponds to an arc from (p, t) to (p, t'), then $f(a)$ is equal to the number of units present at place p in the time period t–t' (possibly overnight).

First of all, this function is a *circulation*. That is, at any vertex v of G one should have:

$$\sum_{a \in \delta^+(v)} f(a) = \sum_{a \in \delta^-(v)} f(a), \qquad (1)$$

the *flow conservation law*. Here $\delta^+(v)$ denotes the set of arcs of G that are entering vertex v and $\delta^-(v)$ denotes the set of arcs of G that are leaving v.

Moreover, in order to satisfy the demand and capacity constraints, f should satisfy the following condition for each arc a corresponding to a stage: $f(a) \leq 5$ and $163f(a) \geq d(a)$.

Here, $d(a)$ is the 'demand' for that stage, that is, the lower bound on the number of seats given in Figure 5 (and 163 is the number of seats of a Type 3 'koploper').

To find the total number of train units used by a given circulation f, one may add up the flow values on the four 'overnight arcs'. So if we wish to minimize the total number of units deployed, we minimize $\sum_{a \in A^\circ} f(a)$.

Here A° denotes the set of overnight arcs. So $|A^\circ| = 4$ in the Amsterdam-Vlissingen example.

It is easy to see that this fully models the problem. Hence determining the minimum number of train units amounts to solving a minimum-cost circulation problem, where the cost function is quite trivial: we have cost$(a) = 1$ if a is an overnight arc and cost$(a) = 0$ for all other arcs.

Having this model, standard minimum-cost flow algorithms give an optimum rolling stock circulation in a fraction of a second. It turns out that for the circulation described above, 22 train units are needed.

It is quite direct to modify and extend the model so as to contain several other problems. Instead of minimizing the number of train units one can minimize the amount of carriage-kilometres that should be made every day, or any linear combination of both quantities.

In addition, one can put an upper bound on the number of units that can be stored at any of the stations.

Instead of considering one line only, one can more generally consider *networks* of lines that share the same stock of railway material, including trains that are scheduled to be split or combined. Nederlandse Spoorwegen has hourly trains from Amsterdam, The Hague and Rotterdam to Enschede, Leeuwarden and Groningen that are combined into one train on a common trajectory. This network is called 'The North-East'.

If only one type of unit is employed for that network, each unit having the same capacity, the problem can be solved quickly even for large networks.

Mixing several types of train units

The problem becomes harder if there are several types of train units of different capacity that can be deployed for the train service and for that NS asked for the help of mathematicians. Clearly, if for each scheduled train we would prescribe the type of unit that should be deployed, the problem could be decomposed into separate problems of the type above. But if we do not make such a prescription and if different types can be coupled together to form a train of mixed composition, we should extend the model to a 'multi-commodity circulation' model.

Let us restrict ourselves to the case Amsterdam-Vlissingen again, where now we can deploy two types of two-way train units that can be coupled together. The two types are type 3, each unit of which consists of 3 carriages, and type 4, each unit of which consists of 4 carriages. Indeed, NS has 'koplopers' of both lengths. Types 3 and 4 have 163 and 218 seats, respectively.

Again, the demands of the train stages are given in Figure 5. The maximum number of carriages that can be in any train is again 15. This means that if a train consists of x units of type 3 and y units of type 4 then $3x+4y \leq 15$ should hold.

It is quite direct to extend the model above to the present case. Again we consider the directed graph $G = (V, A)$ as above. At each arc a, let $f(a)$ be the number of units of type 3 on the stage corresponding to a and let $g(a)$ similarly represent type 4. So both $f : A \longrightarrow \mathbf{Z}_+$ and $g : A \longrightarrow \mathbf{Z}_+$ are circulations, that is, each satisfies the flow circulation law (1). For each stage a, the capacity constraint is now $3f(a) + 4g(a) \leq 15$ and the demand constraint is $163f(a)+218g(a) \geq d(a)$, where again $d(a)$ denotes the number of required seats on stage a (Figure 5).

Let cost_3 and cost_4 represent the cost of purchasing one unit of type 3 and one unit of type 4, respectively. Although train units of type 4 are more expensive than those of type 3, they are cheaper per carriage; that is, $\text{cost}_3 < \text{cost}_4 < \frac{4}{3}\text{cost}_3$. This is due to the fact that engineer's cabins are relatively expensive.

Then we must minimize

$$\sum_{a \in A^\circ} \left(\text{cost}_3 f(a) + \text{cost}_4 g(a)\right).$$

However, there is an important further constraint that makes the problem much harder: it is required that at any of the four stations given (Amsterdam, Rotterdam, Roosendaal and Vlissingen) one may either couple units to or decouple units from a train but not both simultaneously. Moreover, one may couple fresh units only to the front of the train and decouple laid off units only from the rear. So if one must decouple a unit of type 3 from the train, it should be at the rear. Similar rules apply at the terminal stations.

This makes the problem more combinatorial, as the order of the different units in a train does matter. This does not fit directly in the circulation model described above and requires an extension.

To this end, one describes a train composition on a stage a by a vector $z(a)$ in $\mathbf{R}^{C_a}$. Here C_a is the set of compositions that are allowed for stage a. This takes the number of seats required for stage a into account and also the maximum train length. So for a certain stage a, one might have

$$C_a = \{3333, 33333, 3334, 3343, 3433, 4333, 3344, 3434, 4334, 3443, 4343, 4433, 444, 4443, 4434, 4344, 3444\}$$

Here 3433 (for instance) means that the train consists of four units, of types 3, 4, 3 and 3, respectively, seen from the front of the train. Then $z(a)_c$ is equal to 1 for precisely one $c \in C_a$ (namely, the composition c running stage a) and $z(a)_c = 0$ for all other $c \in C_a$. This can be described by integer linear inequalities, and the values of $f(a)$ and $g(a)$ are linear functions of $z(a)$. So we have an integer linear programming model.

A first attempt to include the coupling conditions is to add linear constraints on $z(a)$ and $z(a')$, where a and a' are consecutive stages of a train ride (like Amsterdam-Rotterdam and Rotterdam-Roosendaal of train 2127) so as to exclude transitions that are forbidden by the rule that units should be added only at the front of a train or removed only at the rear of the train and not both at the same stop.

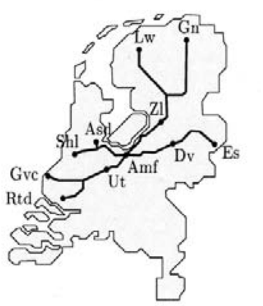

Figure 7 The *North-East*

This however turned out to be computationally infeasible. For those who understand integer linear programming, the linear relaxation of this problem is too loose to obtain good bounds in a branch-and-bound approach.

It took us a long time to realize that by adding even more variables, the problem can be solved relatively fast. For every *allowed* transition, say composition $c \in C_a$ on stage a is allowed to be followed by composition $c' \in C_{a'}$ in the subsequent stage a', we introduce a variable $w_{a,c,a',c'} \geq 0$ and require for every pair of two consecutive stages a and a':

$$z(a)_c = \sum_{c' \in C_{a'}} w_{a,c,a',c'} \text{ for all } c \in C_a \text{ and}$$

$$z(a')_{c'} = \sum_{c \in C_a} w_{a,c,a',c'} \text{ for all } c' \in C_{a'}.$$

This describes all constraints and, importantly, one does not need to require the new variables $w_{a,c,a',c'}$ to be integers. Therefore, the extra variables do not add to the complexity (the branching tree) but rather serve as oil to make the integer programming machinery work. With this, the problem turns out to be solvable with standard integer linear programming software in a few minutes. The number of extra variables is huge but this just helps you to find a solution within a few minutes.

It turns out that for the Amsterdam-Vlissingen problem, one needs 7 units of type 3 and 12 units of type 4. Comparing this solution with the solution for one type only, the possibility of having two types gives both a decrease in the total number of train units and in the total number of carriages needed. The method can be extended to include combinations and splits of trains at intermediate stations and thus also applies to the larger network 'The North-East', where similar savings have been obtained and more passengers can find a seat.

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