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Mathematical connections

It is truly a golden age for mathematics. We are witnessing an unprecedented confluence of fields that demonstrates the unity of mathematical thought. Eugene Wigner has eloquently named this phenomenon 'the unreasonable effectiveness of mathematics'. Mathematical ideas have the remarkable ability of turning up in the most diverse contexts. Paraphrasing the opening of Wigner's essay, whenever one meets a π, a sine or a cosine, one has to ask: where is the circle hidden in my problem?

This deep connectivity is often overlooked by the general public who easily get intimidated or frightened by mathematics. It is a conventional wisdom among publishers of popular science books that every equation cuts the potential readership in half. Most people therefore overlook a modest but crucial ingredient in these equations: the equals sign. In its archetypal form $A = B$, the equals sign connects two worlds represented by A and B. Through it ideas can flow from A to B and back, as if the equals sign conducts the electric current that lights up the 'Aha!' light bulb in the mind indicating the insight gained. Albert Einstein was an absolute master in finding equations with that property. Take $E = mc^2$, which connects mass and energy and is, without a doubt, the most famous equation in the public imagination. The equations of general relativity, although less catchy and well-known, link in an equally surprising and elegant way the worlds of geometry and matter. Another beautiful illustration is given by the Index Theorem of Atiyah and Singer, where the left hand side represents the world of global analysis and partial differential equations, and the right hand side represents the world of geometry and topology. In that sense the discovery of a great theorem can be the mathematical equivalent of the collision of two continents. Just as North and South America bumped into each other three million years ago, and animals and plants started moving up and down along the Panama Isthmus, mathematical ideas can flow between fields if they are connected by a great theorem, generating new forms and finding new applications.

If we pursue the analogy between continental drift and the movement of scientific fields then we are living in a time of exceptional geological activity. Interdisciplinary fields are developing at an unprecedented speed. My own discipline of mathematical physics finds itself in a particularly active period. Recent events illustrate this. The 2006 Fields Medallists Okounkov, Perelman and Werner all used crucial ideas from physics. The 2008 Crafoord Prize was awarded to Fields Medallists Kontsevich and Witten for their work connecting geometry and quantum physics. The new Simons Centre for Geometry and Physics at Stony Brook University, generously endowed by mathematician and financier James Simons, is especially directed toward this active area of research.

The synergy between physics and mathematics is definitely not a new phenomenon. Mathematics has a long history of drawing inspiration from the physical sciences, going back to astrology, architecture and land measurements in Babylonian and Egyptian times. Certainly this reached a high point in the scientific revolution of the 17th century with the development of calculus and classical mechanics. One of its leading architects Galileo has given us the famous image of the 'Book of Nature', which is waiting to be decoded by scientists. In *Il Saggiatore* he writes: 'Philosophy is written in this grand book, the universe, which stands continually open to our gaze. But the book cannot be understood unless one first learns to comprehend the language and read the characters in which it is written. It is written in the language of mathematics, and its characters are triangles, circles, and other geometric figures without which it is humanly impossible to understand a single word of it; without these one is wandering in a dark labyrinth'. And this deep respect for mathematics didn't disappear after the 17th century. At the beginning of the last century we again saw a wonderful intellectual union of physics and mathematics when the great theories of general relativity and quantum mechanics were developed. This was closely watched throughout the mathematical world; mathematicians actively participated, in particular in Göttingen where Hilbert, Minkowksi, Weyl, Von Neumann and many others made important contributions to physics.

Theoretical physics has always been fascinated by the mathematical beauty of its equations. Here we can even quote Feynman, who was certainly not known as a connoisseur of abstract mathematics: 'To those who do not know mathematics it is difficult to get across a real feeling as to the beauty, the deepest beauty, of nature... If you want to learn about nature, to appreciate nature, it is necessary to understand the language that she speaks in'. (Of course, he has also stated: 'If all mathematics disappeared today, physics would be set back exactly one week,' to which a mathematician had the clever answer: 'This was the week that God created the world'.)

But despite the warm feelings expressed by Feynman, the paths of fundamental physics and mathematics started to diverge dramatically in the 1950s and 1960s. In the struggle to understand the stream of new subatomic particles, physicists were close to giving up the hope of an underlying mathematical structure of nature. On the other hand mathematicians were very much in an introspective mode in that period. Dyson stated in his Gibbs Lecture in 1972: 'I am acutely aware of the fact that the marriage between mathematics and physics, which was so enormously fruitful in past centuries, has recently ended in divorce'. But in some sense these were famous last words because at that time the Standard Model of elementary particle physics was born. All the building blocks of that model — gauge fields, curvatures, bundles, covariant derivatives, spinors, Dirac equations — have turned out to have completely natural mathematical interpretations. Soon mathematicians and physicists started to build this dictionary and through the work of Atiyah, Singer, Chern, Yang, 't Hooft, Polyakov and many, many others a new period of fruitful interaction between mathematics and physics was born.

These days, under the influence of quantum theory, the collision of physics and mathematics is producing many new fertile lands. This is particularly true for string theory. Its stimulating influence in mathematics will have a lasting and rewarding impact, whatever its final role in fundamental physics turns out to be. The number of mathematical subfields that come together is dizzying: analysis, geometry, algebra, topology, Lie theory, combinatorics, probability, operator algebras, K-theory, categories — the list goes on and on. One starts to feel sorry for the students who have to learn all this! And it is not only the case that physicists are using advanced mathematical techniques. There are many ideas coming out of physics that have truly influenced pure mathematics. In this respect I like to paraphrase Wigner and speak of 'the unnatural effectiveness of physics in modern mathematics'. To mention just three areas: representation theory, low dimensional topology and algebraic geometry have all been transformed in the last few decades.

In retrospect this development is not as surprising as one might think. A traditional role of physics has been to provide a 'natural' context for mathematical concepts. This has been the case for the classical world but is even more so for the quantum world, which represents a more fundamental description of reality. Even though our intuition, based on everyday experience, is not very well-suited to understanding the properties of elementary particles, ideas from quantum field theory carry tremendous mathematical power. For example, whereas in classical mechanics a particle travels from A to B along a fixed path determined by a variational principle (for example, a geodesic), in quantum mechanics one should consider the 'sum over all histories'; all paths from A to B have to be considered in a weighted ensemble with the classical trajectory being the most likely. Therefore quantum theory naturally leads to summations over objects in certain sets.

A striking example of this principle is mirror symmetry. String theory suggests computations of the number of complex curves on a manifold not degree by degree but in a single stroke by relating the counting function to a classical object (the period of a holomorphic form) on the associated 'mirror manifold', which can have a very different topology. A famous example of the power of mirror symmetry is the original computation of the quintic Calabi-Yau hypersurface in projective four-space by Candelas and collaborators (*Nucl. Phys.* **B359** (1991) 21). This is a hard problem. The 2875 lines (degree one curves) on the quintic are a classical result from the 19th century. The 609250 different conics were only computed around 1980. Finally, the number of twisted cubics 317206375 was the result of a complicated computer program. However, thanks to string theory, we now know the full expansion. Here are the next terms: 242467530000, 229305999987625, 248249742118022000, 295091050570845659250, etc. When considered as coefficients in a power series they form an elegant hypergeometric function.

It is comforting to see how mathematics has been able to absorb the often dizzyingly imprecise heuristics of quantum physics and string theory, and to transform these intuitions into rigorous statements and proofs. Remarkably, these proofs have often *not* followed the path that physical arguments had suggested. It is not the role of mathematicians to clean the dishes of the physicists! On the contrary, in many cases completely new lines of thought had to be discovered in order to find the proofs, as was the case for mirror symmetry.

Niels Bohr was very fond of the notion of complimentarity. Originally this related to the fact that in quantum mechanics an electron could be viewed either as a particle or as a wave, but not both at the same time. The 'correct' point of view — particle or wave — is solely determined by the nature of the question, not by the nature of the electron. In his later years Bohr tried to push this idea to a much more embracing philosophy; one of his favourite complementary pairs was truth and clarity. Perhaps the pair of mathematical rigour and physical intuition should be added as another example of two mutually exclusive qualities. From that perspective, Plato's cave should be updated. Traditionally, this is the place where precise mathematical objects project vague shadows in the physical world. However, in the 'quantum cave' physical theories, which are often far from completely understood, project razor-sharp mathematical shadows and precisely formulated conjectures, which can be checked and proved.

From a wider perspective, beyond the traditional view of theoretical physics, it is clear that mathematics has a unique role to play in the rise of interdisciplinary research. It is ideally positioned to bring disjoint fields of research into contact. There is a charming technical term used in the study of lattices and sphere packings that indicates the number of spheres a given sphere touches: the kissing number (the root lattice of the Lie algebra E8 and the Leech lattice have for example exceptionally high kissing numbers). Perhaps mathematics as a whole can be characterized as a science with an exceptionally high kissing number.

This phenomenon is not only at work within the sciences as a whole but also in mathematics itself, although this is not universally acknowledged and taught to our students. In his powerful essay 'On proof and progress in mathematics' (*Bull. AMS* **30**, 2 (1994) 161–177). Thurston raises the important question of the 'right' definition of a mathematical concept. He illustrates this with the idea of a derivative. Of course there is the familiar $\varepsilon\delta$-definition that scares every first year analysis student. But Thurston raises the issue that there are many other points of view that are equally worthwhile. For example the dimension of time. Everyone has a natural idea about rate and velocity, even near instantaneous — children have no trouble saying 'for a moment I went very fast'. In fact, one can even 'feel' higher derivatives. The second derivative of constant acceleration pushes us deeper into our car seats; the third derivative or 'jerk' induces motion sickness. Equally important of course is the geometric way of thinking, which immediately suggests generalizations to more variables or higher dimensions. Thurston lists many more aspects: linear, infinitesimal, symbolic, approximate. In the end we need all these definitions to bring to life the full concept of a derivative. Deep mathematical notions are therefore like many faceted diamonds. The connection to quantum physics is just another valuable addition.

In this way the essence of mathematics is captured very well by the humble symbol of the equals sign in our equations. This interconnectedness of mathematics, both internally and externally, is perhaps difficult to explain to a general audience, but it is easy to hear! Just pronounce an equation. You will notice that 'is' constitutes the only verb in the sentence. It is an activity, as one has to actively connect A and B. That is an interesting comment on the purpose of this congress and on the nature of mathematics itself.

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